

**$n$ -FOLD GROUPOIDS,  $n$ -TYPES, AND  $n$ -TRACK CATEGORIES**

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ABSTRACT. For each  $n \geq 1$ , we introduce a certain class of  *$n$ -typical  $n$ -fold groupoids* which model all  *$n$ -types* of topological spaces. We use these to provide an algebraic model for  *$n$ -track categories*, that is, categories enriched in  *$n$ -types*. We give an application to the homotopy type of iterated loop spaces.

## INTRODUCTION

Dwyer and Kan have shown that any model category – in fact, any category with a suitable distinguished class of weak equivalences – can be endowed with simplicial function complexes, which capture the same sort of homotopy information as the mapping spaces of **Top** (cf. [DK1, DK2]). They then define a *homotopy theory* to be a simplicially enriched category, and also provide a notion of weak equivalence of homotopy theories, which suggests that there is a “homotopy theory of homotopy theories”, encoded by a suitable model category (cf. [DKS, Be1]).

This idea was further expanded by later work of Rezk, Bergner, Joyal, and others, who provided alternative models for homotopy theories, including the Segal categories of [DKS, §7] and [Be2], the complete Segal spaces of [R], the quasi-categories of Boardman-Vogt (see [J, Lu]), and others. Homotopy theories in this sense are a special case of (weak)  $\infty$ -categories, or  $\omega$ -categories (cf. [KV1, Lei]), sometimes called  *$(\infty, 1)$ -categories*. See the survey in [Be4], which also discusses the passages between the various models.

Note that most homotopy invariants of topological spaces (and other model categories), such as homotopy and (co)homology groups, are graded or filtered by dimension, so it is useful to have “finite-dimensional” approximations to homotopy theories. Let  $\mathcal{S}$  denote the category of simplicial sets, and  $\mathcal{C}$  a simplicially enriched category: applying the  $n$ -Postnikov section functor to the  $\mathcal{S}$ -mapping spaces of  $\mathcal{C}$ , we obtain what is sometimes called an  *$(n, 1)$ -category*. Our goal here is to obtain convenient algebraic models for such categories, from which the invariants of order  $\leq n$  may be easily extracted.

**0.1. Modeling  $n$ -types.** One way to produce algebraic models for  *$(n, 1)$ -categories* is by constructing suitable models of  *$n$ -types*: that is, spaces  $X$  whose homotopy groups  $\pi_k(X, x)$  vanish for  $k > n$ . For this purpose, we need a functor  $\mathcal{Q}_{(n)} : \mathcal{S} \rightarrow \mathcal{G}^n$  into some monoidal category  $(\mathcal{G}^n, \otimes)$ , with the following properties:

- (a) There is a functor  $B : \mathcal{G}^n \rightarrow \mathcal{S}$  such that  $B\mathcal{Q}_{(n)}X$  is weakly equivalent to the  $n$ -th Postnikov section  $P^n X$ .

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*Date:* March 3, 2013.

*1991 Mathematics Subject Classification.* 55S45; 18G50, 18B40.

*Key words and phrases.*  $n$ -type,  $n$ -fold groupoid, track category.

- (b) Both  $\mathcal{Q}_{(n)}$  and  $B$  are strictly monoidal, so they induce functors on the respective enriched categories.
- (c) The functors  $\mathcal{Q}_{(*)}$  are related by suitable Postnikov-type functors  $\Pi_0^{(n)} : \mathcal{G}^n \rightarrow \mathcal{G}^{n-1}$ .
- (d) There is an algebraic formula for extracting the homotopy groups  $\pi_n(BG, x)$  from  $G \in \mathcal{G}^n$ .
- (e) The model specializes to one for  $(k-1)$ -connected  $n$ -types, from which one can obtain the corresponding  $(n-k)$ -type  $k$ -fold loop space (as conjectured in [BD]).
- (f) There is an explicit “algebraic” construction of  $\mathcal{Q}_{(n)}$ .

In connection with the last point, an important desideratum is clearly the effective computability of  $\mathcal{Q}_{(n)}X$  and related invariants, at least in favorable cases.

The basic example for such a setting, in the case  $n = 1$ , is provided by the classical fundamental groupoid functor  $\mathcal{Q}_{(1)} := \hat{\pi}_1$ . Thus, if we let  $P^n\mathcal{S}$  denote the category of  $n$ -types in  $\mathcal{S}$ , then  $\hat{\pi}_1$  induces an equivalence of homotopy categories  $\mathrm{ho} P^1\mathcal{S} \rightarrow \mathbf{Gpd}/\sim$ . The corresponding groupoid-enriched categories are the *track categories* of [BW].

This approach was extended for  $n = 2$  in [BP], where a category  $\mathcal{DbGpd}_t$  of *2-typical double groupoids* was introduced, and a functor  $\mathcal{Q}_{(2)} : \mathcal{S} \rightarrow \mathcal{DbGpd}_t$  was constructed with the above properties. Categories enriched in  $\mathcal{DbGpd}_t$  are called *2-track categories*.

**0.2.  $n$ -typical  $n$ -fold groupoids.** Many structures have been shown to model  $n$ -types of topological spaces: in the path-connected case, these include the  $\mathrm{cat}^n$ -groups of [Lo], the crossed  $n$ -cubes of [ES] and [Po], and the  $n$ -hyper-crossed complexes of [CC].

Special models exist for  $n = 2, 3$ , starting with the crossed modules of [MW], and including the homotopy double groupoids of [BHKP], the homotopy bigroupoids of [HKK], the strict 2-groupoids of [MS], the double groupoids of [CHR], the double groupoids with connections of [BS], the Gray groupoids of [Ler, Be, JT], and the quadratic modules of [Ba1].

In the general case, such models include Batanin’s higher groupoids (see [Bat, C]), Tamsamani’s weak  $n$ -groupoids (see [T, Si]), and the  $n$ -hypergroupoids of [G].

In this paper we introduce a new model for general  $n$ -types, called  *$n$ -typical  $n$ -fold groupoids*. The associated functor  $\mathcal{Q}_{(n)} : \mathcal{S} \rightarrow \mathbf{Gpd}_t^n$  is called the *fundamental  $n$ -fold groupoid* functor, and it is calculated for a Kan complex  $X$  by iterated applications of the ordinary fundamental groupoid  $\mathcal{Q}_{(1)} = \hat{\pi}_1$  to an appropriate  $n$ -fold simplicial space  $\mathrm{or}_{(n)}^* X$ .

We show that  $\mathcal{Q}_{(n)}$  and  $\mathbf{Gpd}_t^n$  enjoy all the properties enumerated above (see Theorem 3.7 and Proposition 3.25), and we prove:

**Theorem A.** *The functors  $\mathcal{Q}_{(n)} : P^n\mathcal{S} \rightarrow \mathbf{Gpd}_t^n$  and  $B : \mathbf{Gpd}_t^n \rightarrow P^n\mathcal{S}$  induce equivalences of homotopy categories.*

[See Theorem 3.27 below].

Within the category  $\mathbf{Gpd}_t^n$  there is a full subcategory  $\mathbf{Gpd}_t^{(n,k)}$  of  $(n, k)$ -typical  $n$ -fold groupoids, and within  $\mathcal{P}^n\mathcal{S}$  we have the subcategories  $\mathcal{P}_k^n\mathcal{S}$  of  $(k-1)$ -connected spaces, and  $\mathcal{P}_{\Omega^k}^n$  of  $n$ -type  $k$ -fold loop spaces. We show:

**Theorem B.** *The functor  $\mathcal{Q}_{(n)}$  restricted to  $\mathcal{P}_k^n\mathcal{S}$  lands in  $\mathbf{Gpd}_t^{(n,k)}$ , and induces an equivalence of homotopy categories between  $\mathrm{ho}\mathcal{P}_k^n\mathcal{S}$  and  $\mathbf{Gpd}_t^{(n,k)}/\sim$ . Furthermore, there is a functor  $BW_{(n,k)} : \mathbf{Gpd}_t^{(n,k)} \rightarrow \mathcal{P}_{\Omega^k}^{n-k}$ , also inducing an equivalence of  $\mathbf{Gpd}_t^{(n,k)}/\sim$  and  $\mathrm{ho}\mathcal{P}_{\Omega^k}^{n-k}$ .*

[See Theorems 5.7 and 5.8 below].

**0.3.  $n$ -track categories.** Since  $n$ -fold groupoids have a cartesian monoidal structure (with the usual cartesian product), and the functor  $\mathcal{Q}_{(n)}$  preserves products, applying  $\mathcal{Q}_{(n)}$  to each mapping space in a simplicially enriched category  $\mathcal{C}$  yields an  $n$ -track category, that is, a category enriched in  $n$ -typical  $n$ -fold groupoids. This is an algebraic model for the  $n$ -Postnikov approximation to  $\mathcal{C}$ ; it applies in particular to the simplicial enrichment of  $\mathcal{C} = \mathbf{Top}$ .

In [BB1] it was shown that the  $E^r$ -term of the homotopy spectral sequence of a simplicial space  $W_\bullet$  depends only its simplicial  $(r-2)$ -Postnikov stem  $\mathcal{P}[r-2]W_\bullet$  (see §5.11 below). Furthermore, when  $W_\bullet$  is obtained by applying a suitable functor  $F$  to a simplicial resolution of a space  $X$ , the  $(r+2)$ -order derived functors of  $F$  applied to  $X$  (as defined in [BB3]) depend only on  $\mathcal{P}[r+2]W_\bullet$ . The same is true in the cosimplicial case: for example, this applies to the (stable or unstable) Adams spectral sequence.

Therefore, a good algebraic model for  $n$ -track categories can be used to compute the higher terms in such spectral sequences (cf. [Ba3]). In the special case  $n = 1$ , this has been carried out in great detail in [Ba2, BJ] for the  $E^3$ -term of the Adams spectral sequence. An important motivation for this paper is to provide a suitable setting for a future extension of this program to the higher terms in the Adams spectral sequence.

**0.4. Organization.** In Section 1 we define the fundamental  $n$ -fold groupoid  $\mathcal{Q}_{(n)}X$  of a space  $X$ . In Section 2 we identify a class of  $n$ -fold groupoids, called  $n$ -typical, and show that the fundamental  $n$ -fold groupoid is of this class. In Section 3 we show that the  $n$ -typical  $n$ -fold groupoids model all  $n$ -types of topological spaces. An alternative approach using Tamsamani's weak  $n$ -groupoids is provided in Section 4. Finally, in Section 5 we describe an application of our model to  $(k-1)$ -connected  $n$ -types and  $k$ -fold loop spaces, and indicate some directions for future applications.

**0.5. Acknowledgements.** This research was supported by the first author's Israel Science Foundation Grant No. 47377, and the second author's Marie Curie International Reintegration Grant No. 256341. The second author would also like to thank the Department of Mathematics at the University of Haifa for its hospitality during a visit in March-April, 2012.

## 1. THE FUNDAMENTAL $n$ -FOLD GROUPOID OF A SPACE

As noted above, the fundamental groupoid  $\hat{\pi}_1 X$  of a (not necessarily connected) space  $X$  is an algebraic model for its 1-type. We now show how the notion of a 2-typical double groupoid defined in [BP] generalizes to all  $n$ .

### 1.1. $n$ -fold simplicial sets.

As in the case of the fundamental groupoid of a topological space  $Y$ , it is easier to construct the  $n$ -fold groupoid from a fibrant simplicial set model, such as the singular simplicial set  $X := SY \in [\Delta^{\text{op}}, \mathbf{Set}]$ . We therefore first recall some notation and constructions for simplicial sets.

**1.2. Definition.** For any category  $\mathcal{C}$ ,  $[\Delta^{\text{op}}, \mathcal{C}]$  is the category of simplicial objects in  $\mathcal{C}$ . We abbreviate  $[\Delta^{\text{op}}, \mathbf{Set}]$  to  $\mathcal{S}$ . If  $\mathcal{C}$  is concrete, the  $n$ -skeleton  $\text{sk}_n X \in [\Delta^{\text{op}}, \mathcal{C}]$  of any  $X \in s\mathcal{C}$  is generated under the degeneracy maps by  $X_0, \dots, X_n$ . The  $n$ -coskeleton functor  $\text{csk}_n : [\Delta^{\text{op}}, \mathcal{C}] \rightarrow [\Delta^{\text{op}}, \mathcal{C}]$  is left adjoint to  $\text{sk}_n$ . We say that  $X$  is  $n$ -coskeletal if the natural map  $X \rightarrow \text{csk}_n X$  is an isomorphism.

For a Kan complex  $X$ , we can use  $\text{csk}_{n+1} X$  as a model for the  $n$ -th Postnikov section  $P^n X$ . For each  $n \geq 0$ , let  $P^n \mathcal{S}$  denote the full subcategory of  $\mathcal{S}$  consisting of simplicial sets  $X$  for which the natural map  $X \rightarrow P^n X$  is a weak equivalence (that is,  $\pi_i(X, x) = 0$  for all  $x \in X$  and  $i > n$ ). An  $n$ -type is (the homotopy equivalence class of) an object in  $P^n \mathcal{S}$ .

For any  $n \geq 0$ , a map  $f : X \rightarrow Y$  in  $\mathcal{S}$  is called an  $n$ -equivalence if it induces isomorphisms  $f_* : \pi_0 X \rightarrow \pi_0 Y$  (of sets), and  $f_\# : \pi_i(X, x) \rightarrow \pi_i(Y, f(x))$  for every  $1 \leq i \leq n$  and  $x \in X_0$ .

**1.3. Notation.** An  $n$ -fold simplicial object in  $\mathcal{C}$  is a functor  $(\Delta^n)^{\text{op}} \rightarrow \mathcal{C}$ , and we denote the category of such by  $[(\Delta^n)^{\text{op}}, \mathcal{C}]$ . Thus  $X \in [(\Delta^n)^{\text{op}}, \mathcal{C}]$  consists of objects  $X_{i_1 i_2 \dots i_n}$  in  $\mathcal{C}$  for each  $n$ -fold multi-index  $i_1 i_2 \dots i_n \in \mathbb{N}^n$ , along with face in degeneracy maps in each of the  $n$  directions, satisfying the usual simplicial identities.

We can identify  $[(\Delta^n)^{\text{op}}, \mathcal{C}]$  with  $[\Delta^{\text{op}}, [(\Delta^{n-1})^{\text{op}}, \mathcal{C}]]$  in  $n$  different ways: that is, we can think of an  $n$ -fold simplicial object  $X \in [(\Delta^n)^{\text{op}}, \mathcal{C}]$  as a simplicial object  $X^{(i)} \in [\Delta^{\text{op}}, [(\Delta^{n-1})^{\text{op}}, \mathcal{C}]]$  for each  $1 \leq i \leq n$ .

More generally, if we choose  $k$  of the  $n$  directions  $1 \leq j_1 < j_2 < \dots < j_k \leq n$ , we obtain a  $k$ -fold simplicial object  $X^{(j_1, j_2, \dots, j_k)}$  over  $[(\Delta^{n-k})^{\text{op}}, \mathcal{C}]$  – i.e.,  $X^{(j_1, j_2, \dots, j_k)} \in [(\Delta^k)^{\text{op}}, [(\Delta^{n-k})^{\text{op}}, \mathcal{C}]]$  is a diagram of objects  $X_{i_{j_1}, \dots, i_{j_k}}^{(j_1, j_2, \dots, j_k)}$  in  $[(\Delta^{n-k})^{\text{op}}, \mathcal{C}]$ . On the other hand, for each object  $\vec{a} \in \Delta^{n-k}$ ,  $X^{(j_1, j_2, \dots, j_k)}(\vec{a}) \in [(\Delta^k)^{\text{op}}, \mathcal{C}]$  is a  $k$ -fold simplicial object in  $\mathcal{C}$ , natural in  $\vec{a}$ .

**1.4. Décalage and ordinal sum.** We first pass to an  $n$ -fold simplicial set, using the functor  $\text{or}_{(n)}^* : [\Delta^{\text{op}}, \mathbf{Set}] \rightarrow [(\Delta^n)^{\text{op}}, \mathbf{Set}]$  induced by the ordinal sum  $\Delta^n \rightarrow \Delta$ : thus  $(\text{or}_{(n)}^* X)_{p_1 \dots p_n} := X_{n-1+p_1+\dots+p_n}$  (cf. [EP, §2]).

Recall that if  $\text{Aug } \mathcal{S}$  is the category of augmented simplicial sets, the functor  $\text{Dec} : \mathcal{S} \rightarrow \text{Aug } \mathcal{S}$  forgets the last face operator (see [I, Du]). This has a right adjoint  $+ : \text{Aug } \mathcal{S} \rightarrow \mathcal{S}$ , which forgets the augmentation, and the associated comonad  $+ \text{Dec}$  is denoted simply by  $\text{Dec} : \mathcal{S} \rightarrow \mathcal{S}$ . This yields a simplicial resolution  $\text{or}_{(2)}^* X \in [(\Delta^2)^{\text{op}}, \mathbf{Set}]$  for any  $X \in \mathcal{S}$ .

Note that  $\text{or}_{(2)}^* X$  is not itself symmetric, but the order-reversing involution on the category  $\Delta$  of finite ordered sets induces an involution taking  $\text{or}_{(2)}^* X$  to its transpose. See Figure 1, where we think of the vertical direction as first, and horizontal as second, and omit the degeneracy maps.

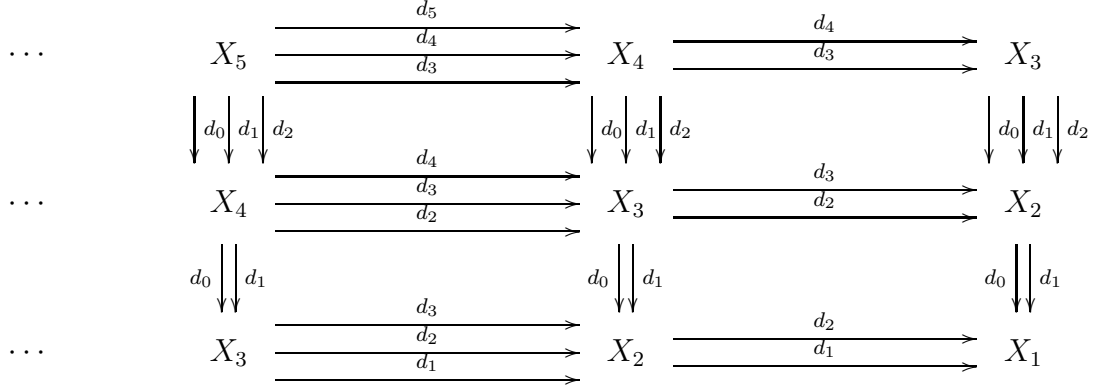


FIGURE 1. Lower right corner of  $\text{or}_{(2)}^* X$

For any presentation of  $\text{or}_{(n)}^* X$  as a simplicial  $(n-1)$ -fold simplicial set (that is, an object in  $[\Delta^{\text{op}}, [(\Delta^{n-1})^{\text{op}}, \mathbf{Set}]]$ ), we see that:

$$(1.5) \quad (\text{or}_{(n)}^* X)_i = \text{or}_{(n-1)}^* \text{Dec}^{i+1} X ,$$

where

$$\text{Dec}^k X := \underbrace{\text{Dec}(\text{Dec} \dots \text{Dec} X \dots)}_k \text{ in } \mathcal{S} = [\Delta^{\text{op}}, \mathbf{Set}] .$$

If we define  $\overline{\text{or}}_{(n-1)}^{*(i)} : [(\Delta^2)^{\text{op}}, \mathbf{Set}] \rightarrow [(\Delta^n)^{\text{op}}, \mathbf{Set}]$  for a bisimplicial set  $X$  by applying  $\text{or}_{(n-1)}^*$  to  $X$  in each simplicial dimension in the  $i$ -th direction ( $i = 1, 2$ ), we have:

$$(1.6) \quad \text{or}_{(n)}^* X = \overline{\text{or}}_{(n-1)}^{*(2)} \text{or}_{(2)}^* X .$$

See Figure 2 for the case  $n = 3$ , where the vertical direction is first, the diagonal is second, and the horizontal is third.

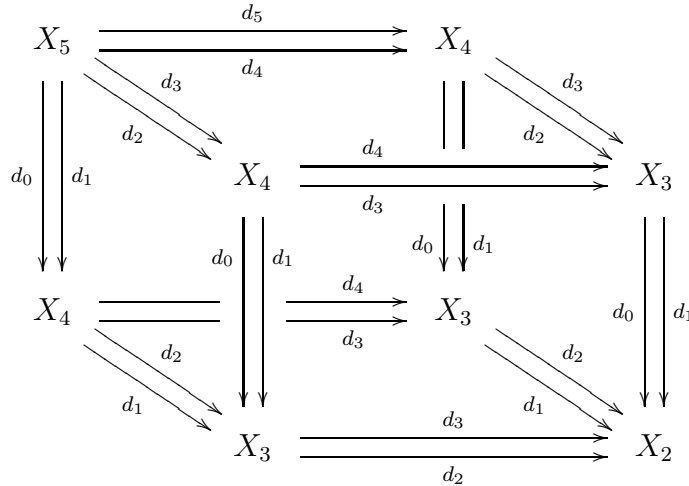


FIGURE 2. Corner of  $\text{or}_{(3)}^* X$

Note that if  $X$  is a fibrant simplicial set, then so is  $\text{Dec } X$ , and the augmentation induces a fibration  $u : \text{Dec } X \rightarrow X$ .

### 1.7. $n$ -fold groupoids and their nerves.

Recall that a *groupoid* is a small category  $G$  in which all morphisms are isomorphisms. It can thus be described by a diagram of sets:

$$(1.8) \quad \begin{array}{ccccc} & & s_0 & & \\ & \curvearrowright & & \curvearrowleft & \\ & d_0 & & i & \\ G_1 \times_{G_0} G_1 & \xrightarrow{c} & G_1 & \xrightarrow{s} & G_0 \\ & \curvearrowleft & & \curvearrowright & \\ & d_2 & & t & \\ & s_1 & & & \end{array}$$

where  $G_0$  is the set of objects of  $G$  and  $G_1$  the set of arrows. Here  $s$  and  $t$  are the source and target functions,  $i$  associates to an object its identity map,  $d_0$  and  $d_2$  are the respective projections, with “inverses”  $s_0$  and  $s_1$ , and  $c$  is the composition – all satisfying the appropriate identities. Let  $\mathbf{Gpd}$  denote the category of groupoids (a full subcategory of the category  $\mathbf{Cat}$  of small categories).

We can think of (1.8) as the 2-skeleton of a simplicial set (with  $G_2 := G_1 \times_{G_0} G_1$ , and  $d_1 = c : G_2 \rightarrow G_1$ ). The *nerve* functor  $\mathcal{N} : \mathbf{Gpd} \rightarrow \mathcal{S}$  (cf. [Se]) assigns to  $G$  the corresponding 2-coskeletal simplicial set  $\mathcal{N}G$ , so:

$$(1.9) \quad (\mathcal{N}G)_n := G_1 \times_{G_0} G_1 \cdots^n G_1 \times_{G_0} G_1$$

for all  $n \geq 2$ , with face maps determined by the associativity of the composition  $c$ .

**1.10. Notation.** If  $\mathcal{V}$  is any category with pullbacks, an *internal groupoid* in  $\mathcal{V}$  is a diagram in  $\mathcal{V}$  of the form (1.8), satisfying the same axioms (see [Bo, I, §8.1]). The category of internal groupoids in  $\mathcal{V}$  is denoted by  $\mathcal{V}(\mathbf{Gpd})$ . Thus an (ordinary) groupoid is an internal groupoid in  $\mathbf{Set}$ .

For each  $n \geq 1$ , an  *$n$ -fold groupoid* is defined inductively to be an internal groupoid in the category  $\mathcal{V} = \mathbf{Gpd}^{n-1}$  of  $(n-1)$ -fold groupoids (where  $\mathbf{Gpd}^0 := \mathbf{Set}$ ), so  $\mathbf{Gpd}^n := \mathbf{Gpd}^{n-1}(\mathbf{Gpd})$ .

If  $X \in [(\Delta^n)^{\text{op}}, \mathbf{Set}]$  is an  $n$ -fold simplicial set, then for each  $1 \leq i \leq n$ ,  $\hat{\pi}_1^{(i)} X$  is the  $(n-1)$ -fold simplicial object in  $\mathbf{Gpd}$  obtained by applying the fundamental groupoid functor  $\hat{\pi}_1$  in the  $i$ -th direction – that is, objectwise to the  $(\Delta^{n-1})^{\text{op}}$ -indexed diagram  $X^{(i)}$ . When  $X$  is fibrant,  $\hat{\pi}_1 X$  has the simple form described in [GJ, I.8].

Similarly, for each  $1 \leq i \leq n$ , let  $\mathcal{N}^{(i)} : \mathbf{Gpd}^n \rightarrow [\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}]$  denote the nerve functor in the  $i$ -th direction. More generally, for any  $k$  of the  $n$  indices  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ ,  $\mathcal{N}^{(i_1, i_2, \dots, i_k)} : \mathbf{Gpd}^n \rightarrow [(\Delta^k)^{\text{op}}, \mathbf{Gpd}^{n-k}]$  takes an  $n$ -fold groupoid  $G$  to a  $k$ -fold simplicial object in  $(n-k)$ -fold groupoids by applying the nerve functor in the indicated  $k$  directions. In particular,  $\mathcal{N}^{(\hat{i})}$  means that we take nerves in all but the  $i$ -th direction.

**1.11. Definition.** The *multinerve*  $\mathcal{N}_{(n)} : \mathbf{Gpd}^n \rightarrow [(\Delta^n)^{\text{op}}, \mathbf{Set}]$  is defined by applying  $\mathcal{N}^{(i)}$  for  $1 \leq i \leq n$  to obtain the  $n$ -fold simplicial set  $\mathcal{N}_{(n)} G := \mathcal{N}^{(1)} \mathcal{N}^{(2)} \dots \mathcal{N}^{(n)} G$ . We say that an  $n$ -fold groupoid  $G$  is *discrete* if  $\mathcal{N}_{(n)} G$  is a constant  $n$ -fold simplicial set.

If  $G \in \mathbf{Gpd}^{n-1}$  is an  $(n-1)$ -fold groupoid, then  $cG$  denotes the  $n$ -fold groupoid which, as a groupoid object in  $\mathbf{Gpd}^{n-1}$ , is discrete on  $G$ . In particular, if  $A$  is a set,  $A_{(n)}^d$  denotes the discrete  $n$ -fold groupoid  $c \cdot \cdots cA$  on  $A$ .

The composite of  $\mathcal{N}_{(n)}$  with the  $n$ -fold diagonal (or homotopy colimit)  $\mathrm{Diag}_{(n)} : [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}] \rightarrow [\Delta^{\mathrm{op}}, \mathbf{Set}]$ , yields the *diagonal nerve* functor  $B := \mathrm{Diag}_{(n)} \mathcal{N}_{(n)}$ , and  $BG$  is called the *classifying space* of  $G$ .

A map of  $n$ -fold groupoids  $f : G \rightarrow G'$  is called a *weak equivalence* if it induces a weak equivalence of simplicial sets  $Bf : BG \rightarrow BG'$ .

### 1.12. The fundamental $n$ -fold groupoid of a space.

We now introduce the central construction of our paper. Its internal analogue in the category of groups is the fundamental  $\mathrm{cat}^n$ -group of a space, due to Bullejos, Cegarra, and Duskin (see [BCD]).

**1.13. Definition.** Let  $\mathcal{Q}_{(n)} : [\Delta^{\mathrm{op}}, \mathbf{Set}] \rightarrow \mathbf{Gpd}^n$  be the composite

$$[\Delta^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathrm{or}_{(n)}^*} [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathcal{P}_{(n)}} \mathbf{Gpd}^n$$

where  $\mathcal{P}_{(n)}$  is left adjoint to  $\mathcal{N}_{(n)} : \mathbf{Gpd}^n \rightarrow [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}]$ . We call  $\mathcal{Q}_{(n)}X$  the *fundamental  $n$ -fold groupoid* of  $X$ .

We shall show that if  $X \in [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}]$  satisfies certain conditions, then  $\mathcal{P}_{(n)}X$  has a particularly simple form. These conditions are satisfied for  $\mathrm{or}_{(n)}^*X$  when  $X$  is fibrant, leading to a simple expression for  $\mathcal{Q}_{(n)}X$ .

**1.14. Definition.** Let  $n \geq 2$ . An  $n$ -fold simplicial set  $X \in [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}]$  is called  *$(n, 2)$ -fibrant* if for each  $1 \leq i \neq j \leq n$ ,  $\vec{a} \in \Delta^{n-2}$ , and  $k = 0, 1, 2$ , the bisimplicial set  $X^{(i,j)}(\vec{a}) \in [(\Delta^2)^{\mathrm{op}}, \mathbf{Set}]$  satisfies:

- (i)  $X^{(i,j)}(\vec{a})_{\bullet k}$  is a 2-coskeletal Kan complex;
- (ii) the two “horizontal” face maps  $d_0, d_1 : X^{(i,j)}(\vec{a})_{1\bullet} \rightarrow X^{(i,j)}(\vec{a})_{0\bullet}$  are fibrations in  $\mathcal{S}$ .

**1.15. Definition.** Let  $G \in [(\Delta^m)^{\mathrm{op}}, \mathbf{Gpd}^{n-m}]$  be an  $m$ -fold simplicial object in  $(n-m)$ -fold groupoids (cf. §1.10). We say that  $G$  is  *$(n, 2)$ -fibrant* if, after applying the nerve functor in each of the  $n-m$  groupoid directions, the resulting  $n$ -fold simplicial set  $\mathcal{N}^{(1,2,\dots,n-m)}G \in [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}]$  is  $(n, 2)$ -fibrant in the sense of Definition 1.14.

We recall the following three results from [BP]:

**1.16. Proposition** ([BP, Prop. 2.10]). *The left adjoint  $\mathcal{P}^{(1)} : [\Delta^{\mathrm{op}}, \mathbf{Gpd}] \rightarrow \mathbf{Gpd}^2$  to the nerve  $\mathcal{N}^{(1)} : \mathbf{Gpd}^2 \rightarrow [\Delta^{\mathrm{op}}, \mathbf{Gpd}]$ , when applied to a  $(2, 2)$ -fibrant simplicial groupoid  $G_{\bullet}$ , is given by  $\hat{\pi}_1^{(1)}G_{\bullet}$  (the functor  $\hat{\pi}_1$  applied in the simplicial direction).*

**1.17. Proposition** ([BP, Prop. 2.11]). *If  $X \in [(\Delta^2)^{\mathrm{op}}, \mathbf{Set}]$  is a  $(2, 2)$ -fibrant bisimplicial set, then  $\hat{\pi}_1^{(1)}X$  is a  $(2, 2)$ -fibrant simplicial groupoid.*

**1.18. Lemma.** *If  $G_{\bullet}$  is a  $(2, 2)$ -fibrant simplicial groupoid (with simplicial sets of objects  $G_{\bullet 0}$  and morphisms  $G_{\bullet 1}$ ), then  $\mathcal{N}^{(2)}\hat{\pi}_1^{(1)}G_{\bullet} = \hat{\pi}_1^{(1)}\mathcal{N}^{(2)}G_{\bullet}$ .*

*Proof.* It suffices to show that for each  $k \geq 2$ :

$$(1.19) \quad \hat{\pi}_1(G_{\bullet 1} \times_{G_{\bullet 0}} \cdots \times_{G_{\bullet 0}} G_{\bullet 1}) \cong \hat{\pi}_1(G_{\bullet 1}) \times_{\hat{\pi}_1(G_{\bullet 0})} \cdots \times_{\hat{\pi}_1(G_{\bullet 0})} \hat{\pi}_1(G_{\bullet 1}) .$$

Since both sides are groupoids, we evidently have equality on objects, and (1.19) holds on morphisms by [BP, App. A, following (8.13)].  $\square$

**1.20. Proposition.** *If  $X \in [(\Delta^n)^{\text{op}}, \mathbf{Set}]$  is  $(n, 2)$ -fibrant, then  $\hat{\pi}_1^{(k)} X$  is  $(n, 2)$ -fibrant.*

*Proof.* By definition of  $(n, 2)$ -fibrancy, for each  $\vec{a} \in \Delta^{n-2}$  and  $1 \leq i \neq j \leq n$ , the bisimplicial set  $X^{(i,j)}()$  satisfies the hypotheses of Proposition 1.17. Hence, applying  $\hat{\pi}_1^{(k)}$  to it yields an  $(n, 2)$ -fibrant object of  $[(\Delta^{n-2})^{\text{op}}, [\Delta^{\text{op}}, \mathbf{Gpd}]]$ .  $\square$

**1.21. Proposition.** *For each  $1 \leq i \leq n$ , the left adjoint*

$$\mathcal{P}^{(i)} : [\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}] \rightarrow \mathbf{Gpd}^n$$

*of*

$$\mathcal{N}^{(i)} : \mathbf{Gpd}^n \rightarrow [\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}] ,$$

*when applied to an  $(n, 2)$ -fibrant simplicial  $(n-1)$ -fold groupoid  $X$ , is given by  $\mathcal{P}^{(i)} X = \hat{\pi}_1^{(i)} X$ .*

*Proof.* We think of the simplicial direction of  $X$  as being the  $i$ -th, and let  $1 \leq j \leq n$  be one of the groupoidal directions (so  $i \neq j$ ). Applying the  $(n-2)$ -fold iterated nerve functor

$$\mathcal{N}^{(ij)} : [\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}] \rightarrow [\Delta^{\text{op}}, [(\Delta^{n-2})^{\text{op}}, \mathbf{Gpd}]] \cong [(\Delta^{n-2})^{\text{op}}, [\Delta^{\text{op}}, \mathbf{Gpd}]]$$

of §1.10 (in all but the  $i$  and  $j$  directions) to  $X$  yields an  $(n-2)$ -fold simplicial object in simplicial groupoids  $\tilde{X}$ . Since  $X$  is  $(n, 2)$ -fibrant, for each  $\vec{a} \in \Delta^{n-2}$ , the simplicial groupoid  $\tilde{X}(\vec{a})$  (see §1.3) satisfies the hypotheses of Proposition 1.16, where the simplicial direction is the original  $i$  and the groupoid direction is the original  $j$ . Using [BP, (8.12)], we can therefore define a composition map:

$$(\mathcal{N}^{(i)} \hat{\pi}_1^{(i)} \tilde{X}(\vec{a}))_1 \times_{(\mathcal{N}^{(i)} \hat{\pi}_1^{(i)} \tilde{X}(\vec{a}))_0} (\mathcal{N}^{(i)} \hat{\pi}_1^{(i)} \tilde{X}(\vec{a}))_1 \longrightarrow (\mathcal{N}^{(i)} \hat{\pi}_1^{(i)} \tilde{X}(\vec{a}))_1 .$$

As the construction is functorial in  $\vec{a} \in (\Delta^{n-2})^{\text{op}}$ , it defines a map in  $\mathbf{Gpd}^{n-1}$ , since it consists of maps in sets commuting with compositions in each of the different directions (see [BP, Appendix A]). Thus  $\hat{\pi}_1^{(i)} X$  is a groupoid object in  $\mathbf{Gpd}^{n-1}$  – that is,  $\hat{\pi}_1^{(i)} X \in \mathbf{Gpd}^n$ .

It remains to show that  $\hat{\pi}_1^{(i)} X = \mathcal{P}^{(i)} X$ . Since the (iterated) nerve functor is fully faithful, again using Proposition 1.16, we see that for any  $n$ -fold groupoid  $Y$  we have natural isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbf{Gpd}^n}(\hat{\pi}_1^{(i)} X, Y) &\cong \text{Hom}_{[(\Delta^{n-2})^{\text{op}}, [\Delta^{\text{op}}, \mathbf{Gpd}]]}(\hat{\pi}_1^{(i)} \tilde{X}, \tilde{Y}) = \\ &= \text{Hom}_{[(\Delta^{n-2})^{\text{op}}, [\Delta^{\text{op}}, \mathbf{Gpd}]]}(\tilde{X}, \mathcal{N}^{(i)} \tilde{Y}) = \text{Hom}_{[\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}]}(X, \mathcal{N}^{(i)} Y) . \end{aligned}$$

Hence  $\hat{\pi}_1^{(i)}$  is left adjoint to  $\mathcal{N}^{(i)}$ , as required.  $\square$

**1.22. Corollary.** *The functor  $\mathcal{Q}_{(n)}$  of §1.13, applied to a Kan complex  $X$ , is:*

$$\mathcal{Q}_{(n)} X = \hat{\pi}_1^{(1)} \hat{\pi}_1^{(2)} \cdots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X .$$



*Proof.* By induction on  $n$ . For  $n = 2$ , see [BP, Corollary 2.12]. Suppose the claim holds for  $n - 1$ . The left adjoint  $\mathcal{P}_{(n)} : [(\Delta^n)^{\text{op}}, \mathbf{Set}] \rightarrow \mathbf{Gpd}^n$  is the composite

$$[(\Delta^n)^{\text{op}}, \mathbf{Set}] \cong [\Delta^{\text{op}}, [(\Delta^{n-1})^{\text{op}}, \mathbf{Set}]] \xrightarrow{\overline{\mathcal{P}}_{(n-1)}} [\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}] \xrightarrow{\mathcal{P}^{(1)}} \mathbf{Gpd}^n,$$

where  $\overline{\mathcal{P}}_{(n-1)}$  is induced by  $\mathcal{P}_{(n-1)}$ , and  $\mathcal{P}^{(1)}$  is left adjoint to the nerve  $\mathcal{N}^{(1)} : \mathbf{Gpd}^n \rightarrow [\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}]$  (cf. §1.10). By (1.5) and the induction hypothesis:

$$(\overline{\mathcal{P}}_{(n-1)} \text{or}_{(n)}^* X)_i = \mathcal{P}_{(n-1)}(\text{or}_{(n)}^* X)_i = \mathcal{P}_{(n-1)} \text{or}_{(n-1)}^* \text{Dec}^{i+1} X,$$

so that, by (1.5) we have:

$$\mathcal{Q}_{(n-1)} \text{Dec}^{i+1} X = \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n-1)}^* \text{Dec}^{i+1} X = \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} (\text{or}_{(n)}^* X)_i.$$

It follows that  $\overline{\mathcal{P}}_{(n-1)} \text{or}_{(n)}^* X = \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X$ . Since  $X$  is a Kan complex,  $\text{or}_{(n)}^* X$  is  $(n, 2)$ -fibrant. Therefore, by Proposition 1.20,  $\hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X$  is  $(n, 2)$ -fibrant. It follows by Proposition 1.21 that

$$\mathcal{P}^{(1)} \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X = \hat{\pi}_1^{(1)} \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X.$$

Therefore,

$$\begin{aligned} \mathcal{Q}_{(n)} X &= \mathcal{P}_{(n)} \text{or}_{(n)}^* X = \mathcal{P}^{(1)} \overline{\mathcal{P}}_{(n-1)} \text{or}_{(n)}^* X \\ &= \mathcal{P}^{(1)} \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X = \hat{\pi}_1^{(1)} \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X, \end{aligned}$$

which concludes the induction step.  $\square$

**1.23. Remark.** The functor  $\text{or}_{(n)}^* : [\Delta^{\text{op}}, \mathbf{Set}] \rightarrow [(\Delta^n)^{\text{op}}, \mathbf{Set}]$  has a right adjoint, a generalized Artin-Mazur codiagonal (cf. [AM, §III] and [BCD, CG]), so both  $\text{or}_{(n)}^*$  and  $\mathcal{P}_{(n)}$  – and thus  $\mathcal{Q}_{(n)}$  – preserve colimits, and in particular coproducts.

On the other hand, clearly  $\text{or}_{(n)}^*$  and  $\hat{\pi}_1$  preserve products when applied to Kan complexes, so  $\mathcal{Q}_{(n)}$  does, too. Therefore,  $\mathcal{Q}_{(n)}$  preserves fiber products over discrete simplicial sets.

## 2. $n$ -TYPICAL $n$ -FOLD GROUPOIDS

In the previous section we defined the fundamental  $n$ -fold groupoid  $\mathcal{Q}_{(n)} X$  of a simplicial set  $X$  (cf. §1.13). When  $X$  is a Kan complex,  $\mathcal{Q}_{(n)} X$  enjoys several useful properties, which are needed to distinguish those  $n$ -fold groupoids which in fact represent  $n$ -types of topological spaces.

### 2.1. Homotopically discrete $n$ -fold groupoids.

We first introduce a higher-dimensional analogue of a groupoid  $G$  whose classifying space is homotopically discrete (that is, a disjoint union of contractible spaces). This will play a central role in defining  $n$ -typical  $n$ -fold groupoids.

**2.2. Definition.** Let  $f : A \rightarrow B$  be a morphism in a category  $\mathcal{C}$  with finite limits. The diagonal map defines a unique section  $s : A \rightarrow A \times_B A$  (so that  $p_1 s = \text{Id}_A = p_2 s$ ,

where  $A \times_B A$  is the pullback of  $A \xrightarrow{f} B \xleftarrow{f} A$  and  $p_1, p_2 : A \times_B A \rightarrow A$  are the two projections). The commutative diagram

$$\begin{array}{ccccc} A \times_B A & \xrightarrow{p_1} & A & \xleftarrow{p_2} & A \times_B A \\ p_2 \downarrow & & f \downarrow & & \downarrow p_1 \\ A & \xrightarrow{f} & B & \xleftarrow{f} & A \end{array}$$

defines a unique morphism  $m : (A \times_B A) \times_A (A \times_B A) \rightarrow A \times_B A$  such that  $p_2 m = p_2 \pi_2$  and  $p_1 m = p_1 \pi_1$ , where  $\pi_1$  and  $\pi_2$  are the two projections. We denote by  $A^f$  the following object of  $\mathcal{C}(\mathbf{Cat})$ :

$$(2.3) \quad (A \times_B A) \times_A (A \times_B A) \xrightarrow{m} A \times_B A \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{p_2} \\ \xleftarrow{s} \end{array} A$$

It easy to see that  $A^f$  is an internal groupoid.

**2.4. Definition.** We define a full subcategory  $\mathbf{Gpd}_{\text{hd}}^n \subset \mathbf{Gpd}^n$  of *homotopically discrete  $n$ -fold groupoids* by induction on  $n \geq 1$ :

A groupoid is called *homotopically discrete* if  $G \cong A^f$  for some map of sets  $f : A \rightarrow B$ . In general, an  $n$ -fold groupoid  $G \in \mathbf{Gpd}^n$  is *homotopically discrete* if  $G \cong A^f$  for some map  $f : A \rightarrow B$  in  $\mathbf{Gpd}_{\text{hd}}^{n-1}$ .

Note that for an (ordinary) groupoid  $G$  this just means that  $\pi_1(BG, x) = 0$  for any  $x \in G_0$ .

**2.5. Example.** Given a commuting diagram of sets:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{\ell} & D \end{array}$$

we obtain a morphism of homotopically discrete groupoids  $v : A^f \rightarrow C^\ell$ . The homotopically discrete double groupoid associated to  $v$  is described in Figure 3, where we abbreviate  $(A \times_B A) \times_{(C \times_D C)} (A \times_B A)$  to  $(A \times_B A) \times_{(g,g)} (A \times_B A)$ , and so on.

$$\begin{array}{ccccc} & & (A \times_B A \times_B A) \times_{(g,g,g)} (A \times_B A \times_B A) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & A \times_B A \times_B A \\ & & \downarrow & & \downarrow \\ (A \times_B A) \times_{(g,g)} (A \times_B A) \times_{(g,g)} (A \times_B A) & \longrightarrow & (A \times_B A) \times_{(g,g)} (A \times_B A) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & A \times_B A \\ \begin{array}{c} \downarrow \downarrow \uparrow \uparrow \end{array} & & \begin{array}{c} \downarrow \downarrow \uparrow \uparrow \end{array} & & \begin{array}{c} \downarrow \downarrow \uparrow \uparrow \end{array} \\ A \times_C A \times_C A & \longrightarrow & A \times_C A & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & A \end{array}$$

FIGURE 3. A homotopically discrete double groupoid

Notice that

$$(A \times_B A) \times_{(g,g)} (A \times_B A) \cong (A \times_C A) \times_{(f,f)} (A \times_C A)$$

via the map  $(a, b, c, d) \mapsto (a, c, b, d)$ , and more generally

$$(A \times_B A) \times_{(g,g)} \cdots \times_{(g,g)} (A \times_B A) \cong (A \times_C A) \times_{(f,f)} \cdots \times_{(f,f)} (A \times_C A)$$

for each  $k \geq 2$ . It follows that

$$(2.6) \quad (N^{(1)}G)_{k-1} = \begin{cases} A^f, & \text{if } k = 1; \\ (A \times_C \cdots \times_C A)^{(f, \dots, f)} & \text{if } k \geq 2 \end{cases}$$

and

$$(2.7) \quad (N^{(2)}G)_{k-1} = \begin{cases} A^g, & \text{if } k = 1; \\ (A \times_B \cdots \times_B A)^{(g, \dots, g)} & \text{if } k \geq 2. \end{cases}$$

Therefore  $(N^{(1)}G)_k$  and  $(N^{(2)}G)_k$  are homotopically discrete groupoids for all  $k \geq 0$ .

Moreover, applying  $\pi_0$  vertically in each column to Figure 3 yields the groupoid  $B^h$ , that is:

$$(2.8) \quad B \times_D B \times_D B \longrightarrow B \times_D B \rightrightarrows B$$

Similarly, applying  $\pi_0$  horizontally in each row yields  $C^\ell$ .

**2.9. Notation.** If  $G \in \mathbf{Gpd}^n$  is an  $n$ -fold groupoid for  $n \geq 2$ , it is a groupoid object in  $(n-1)$ -fold groupoids (cf. §1.10): that is, it is described by a diagram  $G_{1*} \xrightarrow{\cdot} G_{0*}$  in  $\mathbf{Gpd}^{n-1}$ , as in (1.8): it thus has an  $(n-1)$ -fold groupoid of objects denoted by  $G_{0*}$  (which in turn has its  $(n-2)$ -fold groupoid of objects  $G_{00}$  and  $(n-2)$ -fold groupoid of morphisms  $G_{01}$ ). Similarly, the  $(n-1)$ -fold groupoid of morphisms of  $G$  is denoted by  $G_{0*}$ . More explicitly,  $G$  may be described by as diagram in  $\mathbf{Gpd}^{n-2}$  of the form:

$$(2.10) \quad \begin{array}{ccccc} & & G_{11} \times_{G_{10}} G_{11} & \rightrightarrows & G_{01} \times_{G_{00}} G_{01} \\ & & \downarrow c^{1*} & & \downarrow c^{0*} \\ G_{11} \times_{G_{01}} G_{11} & \xrightarrow{c^{1*}} & G_{11} & \xrightleftharpoons[d_0^{*1}]{d_1^{*1}} & G_{01} \\ \parallel & & \downarrow d_1^{1*} & & \downarrow d_1^{0*} \\ G_{10} \times_{G_{00}} G_{10} & \xrightarrow{c^{*0}} & G_{10} & \xrightleftharpoons[d_1^{*0}]{d_0^{*0}} & G_{00} \end{array}$$

More generally for each  $i \geq 2$  we let

$$(2.11) \quad G_{i1} := G_{11} \times_{G_{01}} \cdots \times_{G_{01}} G_{11} \text{ and } G_{i0} := G_{10} \times_{G_{00}} \cdots \times_{G_{00}} G_{10}$$

as limits of  $(n-2)$ -fold groupoids, with  $d_0^{i,*}, d_1^{i,*} : G_{i1} \rightarrow G_{i0}$  induced by the source and target maps.

**2.12. Definition.** Define  $\widehat{\Pi}_0^{(2)} : \mathbf{Gpd}^2 \rightarrow [\Delta^{\text{op}}, \mathbf{Set}]$  on a double groupoid  $G$  to be the result of applying  $\pi_0$  (the coequalizer of the source and target maps of the groupoid) in each simplicial dimension to the simplicial groupoid  $\mathcal{N}^{(2)}G$ . This is equipped with a natural transformation of simplicial groupoids  $\widehat{\gamma}^{(2)} : \mathcal{N}^{(2)}G \rightarrow c\widehat{\Pi}_0^{(2)}G$ , where  $cX$  is the discrete groupoid on a set  $X$  (cf. § 1.11).

For  $n \geq 2$ , we then define  $\widehat{\Pi}_0^{(n)} : \mathbf{Gpd}^n \rightarrow [\Delta^{\text{op}}, \mathbf{Gpd}^{n-2}]$  inductively to be the result of applying  $\widehat{\Pi}_0^{(n-1)}$  in each simplicial dimension to the simplicial  $(n-1)$ -fold groupoid  $\mathcal{N}^{(n)}G$ . This simply means taking coequalizers (in the category of  $(n-2)$ -fold groupoids) of the source and target maps (the bottom vertical maps in (2.10)).

The functor is again equipped with a natural transformation of simplicial  $(n-1)$ -fold groupoids  $\widehat{\gamma}^{(n)} : \mathcal{N}^{(n)}G \rightarrow c\widehat{\Pi}_0^{(n)}G$ , which is  $\widehat{\gamma}^{(n-1)}$  in each simplicial dimension.

We let  $\widehat{\gamma}_{(n)}$  denote the composite

$$G \xrightarrow{\widehat{\gamma}^{(n)}} c\widehat{\Pi}_0^{(n)}G \xrightarrow{c\widehat{\gamma}^{(n-1)}} cc\widehat{\Pi}_0^{(n-1)}\widehat{\Pi}_0^{(n)}G \rightarrow \cdots c \dots c\widehat{\Pi}_0^{(1)} \dots \widehat{\Pi}_0^{(n)}G.$$

2.13. *Remark.* Since  $\pi_0 : \mathbf{Gpd} \rightarrow \mathbf{Set}$  preserves products and coproducts, it preserves fiber products over discrete groupoids. Therefore, the same is true of  $\widehat{\Pi}_0^{(n)}$ .

2.14. **Lemma.** *Let  $G \in \mathbf{Gpd}_{\text{hd}}^n$  be a homotopically discrete  $n$ -fold groupoid. Then:*

- (a) *If  $\mathcal{N}^{(i)} : \mathbf{Gpd}^n \rightarrow [\Delta^{\text{op}}, \mathbf{Gpd}^{n-1}]$  for some  $1 \leq i \leq n$  (cf. §1.10), then  $(\mathcal{N}^{(i)}G)_k$  is homotopically discrete for all  $k \geq 0$ .*
- (b) *The simplicial  $(n-2)$ -fold groupoid  $\widehat{\Pi}_0^{(n)}G$  is the nerve of a homotopically discrete  $(n-1)$ -fold groupoid  $\Pi_0^{(n)}G$ , and  $\widehat{\gamma}^{(n)}$  induces a map of  $n$ -fold groupoids  $\gamma^{(n)} : \mathcal{N}^{(n)}G \rightarrow c\Pi_0^{(n)}G$ .*
- (c) *Each map  $\gamma^{(n)}$  is a weak equivalence; moreover,*

$$c \dots c\Pi_0^{(1)} \dots \Pi_0^{(n)}G \cong G^d := (\pi_0 BG)_{(n)}^d$$

(cf. §1.11), and  $\gamma_{(n)} : G \rightarrow G^d$  induces a weak equivalence:

$$(2.15) \quad \gamma_* : G_1 \times_{G_0} \cdots \times_{G_0}^k G_1 \rightarrow G_1 \times_{G_0^d} \cdots \times_{G_0^d}^k G_1 \quad \text{for all } k \geq 2.$$

*Proof.* By Definition 2.12,  $G$  (as an object of  $\mathbf{Gpd}^2(\mathbf{Gpd}^{n-2})$ ) has the form of Figure 3 for some commuting diagram of  $(n-2)$ -fold groupoids:

$$(2.16) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

(a) By (2.6) and (2.7), the statement holds for  $n = 2$ . Suppose by induction that it holds for  $n-1$ : then  $(\mathcal{N}^{(1)}G)_0 = A^f$  is in  $\mathbf{Gpd}_{\text{hd}}^{n-1}$ . Also  $(\mathcal{N}^{(1)}G)_{k-1} = (A \times_C \cdots \times_C A)^{(f, \dots, f)}$  for  $k \geq 2$ . By definition and the induction hypothesis,

$$(f, \dots, f) : A \times_C \cdots \times_C^k A \longrightarrow B \times_D \cdots \times_D^k B$$

is a morphism in  $\mathbf{Gpd}_{\text{hd}}^{n-1}$ . Hence, by definition,  $(\mathcal{N}^{(1)}G)_{k-1} \in \mathbf{Gpd}_{\text{hd}}^{n-1}$ . Similarly for any  $\mathcal{N}^{(i)}G$ .

(b) As an object of  $\mathbf{Gpd}^2(\mathbf{Gpd}^{n-2})$ ,  $G$  has the form of Figure 3 (as a diagram in  $\mathbf{Gpd}^{n-2}$ ). Therefore, by (a),  $\mathcal{N}^{(1)}\widehat{\Pi}_0^{(n)}G$  is the nerve of the  $(n-1)$ -fold

homotopically discrete groupoid  $\Pi_0^{(n)}G := B^h$ , and the map  $\widehat{\gamma}^{(n)}$  lifts to a map of  $n$ -fold groupoids.

(c) By induction on  $n \geq 2$ . For  $n = 2$ , we saw that  $\Pi_0^{(2)}G = B^h$ , and since each column in Figure 3 is homotopically discrete, we see from (2.8) that the rightmost column is equivalent to  $B$ , the next to  $B \times_D B$ , and so on. Thus  $\mathcal{N}^{(1)}\gamma^{(2)} : \mathcal{N}^{(1)}G \rightarrow \mathcal{N}^{(1)}c\Pi_0^{(2)}G$  induces dimensionwise weak equivalence of simplicial spaces, so a weak equivalence of classifying spaces. Since  $B^h$  is a homotopically discrete groupoid, it is weakly equivalent to  $cD$  (in the notation of (2.16)), which is  $(\pi_0 BG)^d$ .

By (2.6) for each  $k \geq 2$ :

$$G_1 \times_{G_0} \cdots \times_{G_0}^k G_1 = (N^{(1)}G)_{k-1} = (A \times_C \cdots \times_C A)^{(f, \dots, f)} \cong B \times_D \cdots \times_D^k B,$$

while since  $G_1$  is homotopically discrete and  $G_0$  is discrete,  $G_1 \times_{G_0^d} \cdots \times_{G_0^d}^k G_1$  is homotopically discrete, so it is also weakly equivalent to  $B \times_D \cdots \times_D^k B$ . Thus (2.15) holds for  $n = 2$ .

In the induction step,  $\mathcal{N}^{(1)}G$  is a simplicial  $(n-1)$ -fold homotopically discrete groupoid (by (2.6) again), and thus by the induction hypothesis for  $n-1$  we have a weak equivalence

$$(\mathcal{N}^{(1)}\gamma_{(n-1)})_r : (\mathcal{N}^{(1)}G)_r \rightarrow (c \dots c \mathcal{N}^{(1)}\Pi_0^{(2)} \dots \Pi_0^{(n)}G)_r =: P_r$$

in each simplicial dimension  $r \geq 0$ . Applying the  $(n-1)$ -fold nerve  $\mathcal{N}_{(n-1)}$  to both sides, we obtain a map of  $n$ -fold simplicial sets  $\mathcal{N}_{(n)}G \rightarrow P_\bullet$  which is a weak equivalence in each simplicial dimension, so induces a weak equivalence

$$BG := \text{Diag}_{(n)} \mathcal{N}_{(n)}G \rightarrow \text{Diag}_{(n)} P_\bullet.$$

However,  $P_\bullet$  is discrete in all but the first simplicial direction, where it is (the nerve of) a homotopically discrete groupoid  $H := \Pi_0^{(2)} \dots \Pi_0^{(n)}G$ . In fact,  $H = (B^d)^{h^d}$ , in the notation of §2.2, where  $h^d : B^d \rightarrow D^d$  is the discretization of the map  $h : B \rightarrow D$  in (2.16).

Therefore,  $\text{Diag}_{(n)} P_\bullet = BH$  has  $\pi_0 BH = \pi_0 H^d = \pi_0 BG$  while  $\pi_i BH = 0$  for  $i \geq 1$ , and the map  $\gamma_{(n)} = \gamma^{(1)} \circ \gamma_{(n-1)}$  induces the requisite weak equivalence. Since also  $\gamma_{(n)} = \gamma_{(n-1)} \circ \gamma^{(n)}$ , we deduce by induction that  $\gamma^{(n)}$  is a weak equivalence, too.

To show (2.15), note that by (2.8) we have:

$$(\Pi_0^{(n)}G)_2 = \Pi_0^{(n-1)}(G_1 \times_{G_0} G_1) = B \times_D B \times_D B = (B \times_D B) \times_B (B \times_D B),$$

which by the induction hypothesis (2.15) and Remark 2.13 equals:

$$\Pi_0^{(n-1)}G_1 \times_{\Pi_0^{(n-1)}G_0} \Pi_0^{(n-1)}G_1 \simeq \Pi_0^{(n-1)}G_1 \times_{(\Pi_0^{(n-1)}G_0)^d} \Pi_0^{(n-1)}G_1 = \Pi_0^{(n-1)}(G_1 \times_{G_0^d} G_1).$$

That is, we have a commuting square

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{\gamma^{(n-1)}} & \Pi_0^{(n-1)}(G_1 \times_{G_0} G_1) \\ \gamma_* \downarrow & & \downarrow \simeq \\ G_1 \times_{G_0^d} G_1 & \xrightarrow{\gamma^{(n-1)}} & \Pi_0^{(n-1)}(G_1 \times_{G_0^d} G_1) \end{array}$$

in which three of the maps are weak equivalences, so  $\gamma_*$  is, too.

Similarly for all  $k > 2$ . □

### 2.17. $n$ -typical $n$ -fold groupoids.

We now present the central notion of this paper: a class of  $n$ -fold groupoids which will be shown (in Section 3) to model all  $n$ -types of topological spaces.

**2.18. Definition.** An  $n$ -fold groupoid  $G$  is said to be *symmetric* if for each  $1 \leq i < j \leq n$  and  $\vec{a} \in \Delta^{n-2}$ , the double groupoid  $H := G(\vec{a})$  has isomorphisms (natural in  $\vec{a}$ )  $H_{0*} \cong H_{*0}$  and  $H_{1*} \cong H_{*1}$ .

This implies in particular that for any of the  $n(n-1)$  ways of identifying  $G$  as an object of  $\mathbf{Gpd}^2(\mathbf{Gpd}^{n-2})$ , the two directions (horizontal and vertical) in (2.10) are isomorphic diagrams in  $\mathbf{Gpd}^{n-2}$ . In particular, we can abbreviate either to  $G_1 \xrightarrow{\gamma} G_0$ .

**2.19. Definition.** For each  $n \geq 1$ , the full subcategory  $\mathbf{Gpd}_t^n$  of  $\mathbf{Gpd}^n$ , whose objects are called  *$n$ -typical  $n$ -fold groupoids*, is defined by induction on  $n$ , as follows:

For  $n = 1$ , any groupoid is 1-typical; suppose we have defined  $\mathbf{Gpd}_t^{n-1}$ . We say that a symmetric  $n$ -fold groupoid  $G = (G_1 \xrightarrow{\gamma} G_0)$  is  *$n$ -typical* if

- (i)  $G_0 := G_0^{(n)}$  is in  $\mathbf{Gpd}_{\text{hd}}^{n-1}$ ;
- (ii)  $G_1 := G_1^{(n)}$  is in  $\mathbf{Gpd}_t^{n-1}$ , and for each  $k \geq 2$ ,  $G_1 \times_{G_0} \cdots \times_{G_0}^k G_1$  is in  $\mathbf{Gpd}_t^{n-1}$ ;
- (iii) The functor  $\widehat{\Pi}_0^{(n)}$  of §2.12 lifts to a functor  $\Pi_0^{(n)} : \mathbf{Gpd}_t^n \rightarrow \mathbf{Gpd}_t^{n-1}$ .
- (iv) The map

$$G_1 \times_{G_0} \cdots \times_{G_0}^k G_1 \longrightarrow G_1 \times_{G_0^d} \cdots \times_{G_0^d}^k G_1$$

induced by  $\gamma : G \rightarrow G_0^d$  yields a weak equivalence on classifying spaces for all  $k \geq 2$ .

**2.20. Remark.** For  $n = 2$ , the above definition is slightly more general than [BP, Definition 2.21]. In fact, in [BP] both maps  $G_1 \xrightarrow{\gamma} G_0$  are required to be fibrations of groupoids, and this implies conditions (iii)-(iv).

**2.21. The iterated arrow object.** For any  $n$ -fold (symmetric) groupoid  $G$  and  $1 \leq k \leq n$ , we define its  *$k$ -fold object of arrows* to be the  $(n-k)$ -fold groupoid:

$$G_{1 \dots 1_k}^{(n-k+1 \dots n)} := \mathcal{W}_{(n,k)} G.$$

Note that by Definition 2.19(ii), if  $G$  is  $n$ -typical,  $\mathcal{W}_{(n,1)} G$  is  $(n-1)$ -typical, so by induction we have a functor  $\mathcal{W}_{(n,k)} : \mathbf{Gpd}_t^n \rightarrow \mathbf{Gpd}_t^{n-k}$ , since

$$(2.22) \quad \mathcal{W}_{(n,k)} = \mathcal{W}_{(n-k+1,1)} \mathcal{W}_{(n-k+2,1)} \cdots \mathcal{W}_{(n-1,1)} \mathcal{W}_{(n,1)}.$$

We recall the following notion and fact from [BP]:

**2.23. Definition.** A map  $f : W \rightarrow V$  of bisimplicial sets is called a *diagonal  $n$ -equivalence* if  $f_k^h : W_k^h \rightarrow V_k^h$  is an  $(n-k)$ -equivalence for each  $k \leq n$ .

**2.24. Proposition** ([BP, Prop. 3.9]). *If  $f : W \rightarrow V$  is a diagonal  $n$ -equivalence, then the induced map  $\text{Diag } f : \text{Diag } W \rightarrow \text{Diag } V$  is an  $n$ -equivalence.*

**2.25. Lemma.** *For any  $G \in \mathbf{Gpd}_t^n$ , the map  $\widehat{\gamma}^{(n)} : \mathcal{N}^{(n)}G \rightarrow c\widehat{\Pi}_0^{(n)}G$  of Definition 2.12 lifts to an  $(n-1)$ -equivalence of  $n$ -fold groupoids  $\gamma^{(n)} : G \rightarrow c\Pi_0^{(n)}G$ .*

*Proof.* Clearly the map  $\widehat{\gamma}^{(n)}$  lifts to a map of  $n$ -fold groupoids. We show that this is an  $(n-1)$ -equivalence by induction on  $n$ . It is clear for  $n = 1$ . Suppose, inductively, it holds for  $n - 1$ .

By construction we have

$$(\Pi_0^{(n)}G)_r := (\mathcal{N}^{(n)}\Pi_0^{(n)}G)_r^{(n)} = \Pi_0^{(n-1)}((\mathcal{N}^{(n)}G)_r^{(n)}) ,$$

and therefore, for each  $r \geq 0$  there is a map

$$(\mathcal{N}^{(n)}\gamma^{(n-1)})_r : (\Pi_0^{(n)}G)_r \rightarrow (c\Pi_0^{(n-1)}G)_r .$$

By taking realizations, we obtain a map of simplicial spaces  $B\gamma^{(n-1)}$ . We claim that this map, thought of as a map of bisimplicial sets, is a diagonal  $(n-1)$ -equivalence (cf. §2.23). In fact, since  $G_0 = (\mathcal{N}^{(n)}G)_0^{(n)}$  is homotopically discrete, by Lemma 2.14,  $(B\gamma^{(n-1)})_0$  is a weak equivalence, hence in particular an  $(n-1)$ -equivalence. By the induction hypothesis  $(B\gamma^{(n-1)})_r$  is a  $(n-2)$ -equivalence for all  $r \geq 1$ . Hence by Proposition 2.24,  $B\gamma^{(n-1)}$  is an  $(n-1)$ -equivalence.  $\square$

**2.26. Remark.** From Lemmas 2.14 and 2.25 we see that a symmetric homotopically discrete  $n$ -fold groupoid is  $n$ -typical.

### 3. $n$ -TYPES

In this section we prove the main result of this paper, Theorem 3.27, which asserts that  $n$ -typical  $n$ -fold groupoids model all  $n$ -types.

#### 3.1. The homotopy type of an $n$ -typical $n$ -fold groupoid.

We start by showing that if  $G \in \mathbf{Gpd}_t^n$ , then its diagonal nerve  $BG$  (cf. §1.10) is an  $n$ -type; that is,  $\pi_i(BG, x) = 0$  for all  $x \in BG$  and  $i > n$ . We prove this using a spectral sequence computation of  $\pi_i(BG, x)$ . In Section 4, we give a second proof using Tamsamani's weak  $n$ -groupoids.

In [Q2], Quillen constructed a spectral sequence for a bisimplicial group, which was generalized in [BF] to define the *Bousfield-Friedlander spectral sequence* of a bisimplicial set  $X_{\bullet\bullet}$ , with

$$(3.2) \quad E_{s,t}^2 = \pi_s^h \pi_t^v X_{\bullet\bullet} \Rightarrow \pi_{s+t} \text{Diag } X_{\bullet\bullet} .$$

See [DKSt, §8.4] for an alternative construction when  $X_{\bullet\bullet}$  is connected in each simplicial dimension. The spectral sequence need not converge otherwise; however, we have the following *sufficient* condition for convergence (cf. [BF, B.3]):

**3.3. Definition.** Think of a bisimplicial set  $X_{\bullet\bullet} \in [(\Delta^2)^{\text{op}}, \mathbf{Set}]$  as a (horizontal) simplicial object in  $\mathcal{S}$  (with the simplicial direction inside  $\mathcal{S}$  thought of as being vertical). In this notation, a  $k$ - $\pi_t$ -*matching collection* at  $a \in X_{n,0}$  (for  $0 \leq k \leq n$ ) is a set of elements  $x_i \in \pi_t(X_{n-1,\bullet}, d_i^h a)$  ( $0 \leq i \leq n, i \neq k$ ), such that:

$$(3.4) \quad (d_i^h)_* x_j = (d_{j-1}^h)_* x_i$$

for every  $(0 \leq i < j \leq n, i, j \neq k)$ .

We say that  $X_{\bullet\bullet}$  satisfies the  $\pi_*$ -Kan condition if for every  $n, t \geq 1$ ,  $0 \leq k \leq n$ ,  $a \in X_{n,0}$ , and  $k$ - $\pi_t$ -matching collection  $(x_i)_{i \neq k}^n$  at  $a$ , there is a fill-in  $w \in \pi_t^v(X_{n\bullet}, a)$  such that  $(d_i^h)_* w = x_i$  for all  $0 \leq i \leq n$  ( $i \neq k$ ).

By [BF, Theorem B.5], if  $X_{\bullet\bullet}$  satisfies the  $\pi_*$ -Kan condition – for example, if each  $X_{n\bullet}$  is connected – then the spectral sequence (3.2) converges.

**3.5. Notation.** Note that for any simplicial set  $Y$  and  $t \geq 1$ , the  $t$ -th homotopy group  $\pi_t(Y, y)$ , as  $y \in Y$  varies, constitutes a *semi-discrete groupoid*, in the sense of [BP, §1] – that is, a disjoint union of groups (abelian, if  $t \geq 2$ ). We denote it by  $\hat{\pi}_t Y$ .

**3.6. Lemma.** *Let  $G_\bullet \in [\Delta^{\text{op}}, \mathbf{Gpd}]$  be a groupoid in  $\mathcal{S}$  satisfying (2.15), with  $G_0$  homotopically discrete. Then the bisimplicial set  $X_{\bullet\bullet} := \mathcal{N}G_\bullet$  satisfies the  $\pi_*$ -Kan condition, and for each  $t \geq 1$ ,  $\hat{\pi}_t X_{\bullet\bullet}$  is a groupoid object in semi-discrete groupoids, so is 2-coskeletal.*

*Proof.* We think of the simplicial direction as vertical. Let  $X_k = (\mathcal{N}G_\bullet)_k$ . Since  $X_0$  is homotopically discrete (that is, a disjoint union of contractible spaces), the groupoid  $\hat{\pi}_t X_0$  is discrete, so any  $k$ - $\pi_t$ -matching collection for  $n = 1$  is trivial.

For  $n = 2$ , note that  $X_2 = X_1 \times_{X_0} X_1$ , so any  $a \in X_{2,0}$  is of the form  $a = (a', a'')$ , where  $d_1 a' = d_0 a'' =: b$ . Moreover,  $d_0 a = a'$ ,  $d_1 a = a' \star a''$  (where  $\star$  denotes the groupoid composition), and  $d_2 a = a''$ .

Thus for  $t \geq 1$ , there are three cases for a  $k$ - $\pi_t$ -matching collection  $(x_i \in \pi_t^v(X_1, d_i a))_{i \neq k}$  at  $a$ :

- (i) When  $k = 1$ , the fill-in  $w \in \pi_t^v(X_2, a)$  for  $x_0$  and  $x_2$  is the pull-back pair  $(x_0, x_2)$  in

$$\pi_t^v(X_2, a) = \pi_t^v(X_1, a') \times_{\pi_t^v(X_0, b)} \pi_t^v(X_1, a'').$$

- (ii) When  $k = 0$ , the fill-in  $w = (y, x_2)$  for  $x_1$  and  $x_2$  should satisfy  $x_1 = d_1 w = y \star x_2$ , so  $y = x_1 \star (x_2)^{-1}$ , using the groupoid structure on  $\hat{\pi}_t^v X_1$ .

- (iii) The case  $k = 2$  is similar.

For  $n > 2$  the proof of the  $\pi_*$ -Kan condition is analogous; however, because  $\hat{\pi}_t X_{\bullet\bullet}$  is 2-coskeletal, we do not even need to verify it, since the spectral sequence (3.2) from the  $E^2$ -term on then depends only on the 2-truncation of  $X_{\bullet\bullet}$  in the horizontal direction.  $\square$

**3.7. Theorem.** *For any  $n$ -typical  $n$ -fold groupoid  $G \in \mathbf{Gpd}_t^n$ ,  $BG$  is an  $n$ -type, and for each base point  $x_0 \in G_{0\dots 0}$  we have:*

$$(3.8) \quad \pi_k(BG; x_0) \cong \begin{cases} \mathcal{W}_{(n,n)}(x_0, x_0) & \text{if } k = n \\ \mathcal{W}_{(n,n-k)}(\Pi_0^{(k+1)} \dots \Pi_0^{(n)})(x_0, x_0) & \text{if } k < n. \end{cases}$$

Here  $\mathcal{W}_{(n,n)}G(a, b)$  (cf. §2.21) is the set of morphisms from  $a$  to  $b$  in the groupoid  $\mathcal{W}_{(n,n-1)}G$  (in any of the  $n$  isomorphic directions) so in particular  $\mathcal{W}_{(n,n)}G(a, a)$  is the group of automorphisms of  $a$  (which is abelian for  $n \geq 2$ ).

*Proof.* Since  $BG$  is  $\text{Diag}_{(n)} \mathcal{N}_{(n)}G$ , we show this by induction on the diagonals and nerves:



For each  $\vec{\mathbf{a}} \in \Delta^{n-2}$ , the double groupoid  $G^{(1,2)}(\vec{\mathbf{a}}) \in \mathbf{Gpd}^2$  (in the notation of §1.3) satisfies the hypotheses of Lemma 3.6, by Definition 2.19. Therefore, the Bousfield-Friedlander spectral sequence for the bisimplicial set  $X(\vec{\mathbf{a}}) := \mathcal{N}^{(1,2)}G$  converges to  $\pi_*(\text{Diag } X(\vec{\mathbf{a}}))$ . Moreover,  $\pi_t^v X(\vec{\mathbf{a}})$  is 2-coskeletal for each  $t \geq 1$ , by the Lemma, as is  $\pi_0^v X(\vec{\mathbf{a}})$  (by Definition 2.19 again). Thus in the  $E^2$ -term of the spectral sequence only the two right columns of two bottom rows can be non-zero, so that  $\text{Diag } X(\vec{\mathbf{a}})$  is a 2-type. In fact, the rightmost column is zero (except at the bottom), so we can read off the homotopy groups of  $\text{Diag } X(\vec{\mathbf{a}})$  from those of  $X(\vec{\mathbf{a}})$ .

Since  $\text{Diag}$  is functorial in  $\vec{\mathbf{a}} \in (\Delta^{n-2})^{\text{op}}$ , we see that the resulting  $(n-1)$ -simplicial object  $Y := \text{Diag}^{(1,2)} \mathcal{N}^{(1,2)}G$  is in  $[\Delta^{\text{op}}, \mathbf{Gpd}^{n-2}]$ , with each  $Y(\vec{\mathbf{a}}) \in \mathcal{S}$  a 2-type (in the simplicial, i.e., vertical, direction). Since  $G_0$  was a homotopically discrete  $(n-1)$ -fold groupoid,  $Y_0^v$  (in the simplicial direction) is a homotopically discrete  $(n-2)$ -fold groupoid. Moreover, for any choice of a third (groupoid) direction  $i$ , and each  $\vec{\mathbf{b}} \in \Delta^{n-3}$ , by Definition 2.19, the bisimplicial groupoid  $Z_{\bullet\bullet} := \mathcal{N}^{(1,2)}G^{(1,2,i)}(\vec{\mathbf{b}})$  has a weak equivalence of bisimplicial sets

$$Z_{\bullet\bullet k} = Z_{\bullet\bullet 1} \times_{Z_{\bullet\bullet 0}} \cdots \times_{Z_{\bullet\bullet 0}} Z_{\bullet\bullet 1} \xrightarrow{\simeq} Z_{\bullet\bullet 1} \times_{G_0^d} \cdots \times_{G_0^d} Z_{\bullet\bullet 1}$$

for each  $k \geq 2$ , natural in  $\vec{\mathbf{b}}$  (note that  $G^d$  is independent of  $\vec{\mathbf{b}}$ ). This map therefore induces a weak equivalence on diagonals (homotopy colimits, in the bisimplicial direction). Thus each simplicial groupoid  $Y(\vec{\mathbf{b}}) = \text{Diag } Z_{\bullet\bullet}$  satisfies the hypotheses of Lemma 3.6.

Now assume by descending induction on  $2 \leq k < n$  that we have  $Y \in [\Delta^{\text{op}}, \mathbf{Gpd}^{n-k}]$ , with  $Y(\vec{\mathbf{a}}) \in \mathcal{S}$  a  $k$ -type for each  $\vec{\mathbf{a}} \in \Delta^{n-k}$ , with  $Y_0^v$  a homotopically discrete  $(n-k)$ -fold groupoid. Here the first (vertical) direction is simplicial. For any choice of a second (groupoid) direction, and each  $\vec{\mathbf{b}} \in \Delta^{n-k-1}$ , the simplicial groupoid  $Y^{(1,2)}(\vec{\mathbf{b}}) \in [\Delta^{\text{op}}, \mathbf{Gpd}]$  satisfies the hypotheses of Lemma 3.6. Therefore, (3.2) converges, with only the two right columns of the bottom  $k$  rows non-zero, and  $\text{Diag } Y(\vec{\mathbf{a}})$  is thus a  $(k+1)$ -type. When  $k = n-1$ ,  $Y$  is a simplicial groupoid which is an  $(n-1)$ -type in the simplicial direction, with  $BG$  appearing as  $\text{Diag } Y$ .

For any 2-typical double groupoid  $G$ , the  $E^2$ -term of the Bousfield-Friedlander spectral sequence for the bisimplicial set  $X_{\bullet\bullet} = \mathcal{N}^h \mathcal{N}^v G$  survives to  $E^\infty$ . Moreover, because  $G_0$  is homotopically trivial,  $E_{1,0}^2 = \pi_1 \pi_0 X_{\bullet\bullet} = 0$ , so in fact by Lemma 3.6

$$\pi_i(\text{Diag } X_{\bullet\bullet}, x_0) = \begin{cases} E_{0,0}^2 = \pi_0 \pi_0(X_{\bullet\bullet}, x_0) & \text{if } i = 0 \\ E_{0,1}^2 = \pi_0 \pi_1(X_{\bullet\bullet}, x_0) & \text{if } i = 1 \\ E_{1,1}^2 = \pi_1 \pi_1(X_{\bullet\bullet}, x_0) & \text{if } i = 2, \end{cases}$$

for each choice of a base-point  $x_0$  in  $G_{00}$ . In fact,  $\pi_1 \pi_1(X_{\bullet\bullet}, x_0)$  is just the automorphism group of  $G_1$ , i.e.,  $\mathcal{W}_{(2,2)}G(x_0, x_0)$

Therefore, given an  $n$ -typical  $n$ -fold groupoid  $G$ , by what we have shown above we see that

$$\pi_n(BG; x_0) \cong \mathcal{W}_{(n,n)}G(x_0, x_0)$$

for each  $x_0 \in G_{0,\dots,0}$  Moreover, by Lemma 2.25 we have

$$\pi_i(BG, x_0) \cong \pi_i(B\Pi_0^{(n-k+1)} \dots \Pi_0^{(n)} G, x_0)$$

for all  $0 \leq i \leq n-k$ , and  $\Pi_0^{(n-k+1)} \dots \Pi_0^{(n)} G$  is an  $(n-k)$ -typical  $(n-k)$ -fold groupoid, so in particular (3.8) holds for each  $0 \leq k \leq n$ .  $\square$

**3.9. Remark.** Theorem 3.7 provides an algebraic expression for the homotopy groups of  $BG$  in terms of the  $n$ -fold groupoid  $G$  itself; hence it yields an intrinsic (algebraic) definition of the notion of weak equivalences among  $n$ -typical  $n$ -fold groupoids.

### 3.10. Modeling $n$ -types.

In the remaining part of this section we prove that  $n$ -typical  $n$ -fold groupoids model  $n$ -types. We need a preliminary lemma which gives a more transparent description of the fundamental  $n$ -fold groupoid functor  $\mathcal{Q}_{(n)} = \mathcal{P}_{(n)} \text{or}_{(n)}^*$ , (Definition 1.13).

**3.11. Remark.** Given a Kan complex  $X$ , we have a natural fibration of simplicial sets  $u : \text{Dec } X \rightarrow X$  (cf. §1.1), and the resulting internal groupoid  $(\text{Dec } X)^u \in \mathbf{Gpd}[\Delta^{\text{op}}, \mathbf{Set}]$  of §2.2. If we let

$$(3.12) \quad \mathcal{L}_{(k)} X := \begin{cases} \text{Dec } X & \text{if } k = 1 \\ \text{Dec } X \times_{X \cdot \dots \times X}^k \text{Dec } X & \text{if } k \geq 2 \end{cases}$$

we see that  $(\mathcal{N}(\text{Dec } X)^u)_{k-1} = \mathcal{L}_{(k)} X$  for  $k \geq 1$ , with

$$(3.13) \quad \mathcal{L}_{(k)} X = \mathcal{L}_{(2)} X \times_{\text{Dec } X \cdot \dots \times_{\text{Dec } X}^{k-1} \text{Dec } X} \mathcal{L}_{(2)} X$$

for all  $k \geq 1$ .

If  $X$  is reduced,  $\text{Dec } X$  is contractible, so  $\mathcal{L}_{(2)} X$  models the loop space  $\Omega X$ . In general,  $\mathcal{L}_{(2)} X$  is homotopy equivalent to  $\mathbb{P}X$  of [Du, §2.2].

**3.14. Lemma.** *Let  $X$  be a Kan complex.*

(a) *For each  $k \geq 0$ :*

$$(3.15) \quad (\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X)_k = \mathcal{Q}_{(n-1)}(\mathcal{L}_{(k+1)} X) .$$

*Therefore, for  $k \geq 2$ :*

$$(3.16) \quad \mathcal{Q}_{(n-1)}(\mathcal{L}_{(k+1)} X) \cong \mathcal{Q}_{(n-1)} \mathcal{L}_{(2)} X \times_{\mathcal{Q}_{(n-1)} \text{Dec } X \cdot \dots \times_{\mathcal{Q}_{(n-1)} \text{Dec } X}^k \mathcal{Q}_{(n-1)} \text{Dec } X} \mathcal{Q}_{(n-1)} \mathcal{L}_{(2)} X .$$

(b) *If  $X$  is homotopically discrete, then:*

$$(3.17) \quad \mathcal{Q}_{(n)}(\mathcal{L}_{(k)} X) \cong \mathcal{Q}_{(n)} \text{Dec } X \times_{\mathcal{Q}_{(n)} X \cdot \dots \times_{\mathcal{Q}_{(n)} X}^k \mathcal{Q}_{(n)} \text{Dec } X} \mathcal{Q}_{(n)} \text{Dec } X .$$

*Proof.* (a) We will show that for  $n \geq 2$ :

$$(3.18) \quad \mathcal{N}^{(n)} \mathcal{Q}_{(n)} X = \overline{\mathcal{Q}}_{(n-1)}^{(2)} \mathcal{N}^{(2)} \hat{\pi}_1^{(2)} \text{or}_{(2)}^* X ,$$

where  $\overline{\mathcal{Q}}_{(n-1)}^{(2)}$  is obtained by applying  $\mathcal{Q}_{(n-1)}$  in each simplicial dimension in the second direction to the bisimplicial object  $\mathcal{N}^{(2)} \hat{\pi}_1^{(2)} \text{or}_{(2)}^*$ .

From (1.6) we see that that:

$$\overline{\text{or}}_{(n-1)}^*(2) \mathcal{N}^{(2)} \hat{\pi}_1^{(2)} \text{or}_{(2)}^* X = \mathcal{N}^{(n)} \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X$$

(see Figures 1-2 for the case  $n = 3$ ), and since:

$$\overline{\mathcal{Q}}_{(n-1)}^{(2)} \mathcal{N}^{(2)} \hat{\pi}_1^{(2)} \text{or}_{(2)}^* X = \hat{\pi}_1^{(1)} \dots \hat{\pi}_1^{(n-1)} \overline{\text{or}}_{(n-1)}^{*(2)} \mathcal{N}^{(2)} \hat{\pi}_1^{(2)} \text{or}_{(2)}^* X$$

we deduce that:

$$(3.19) \quad \overline{\mathcal{Q}}_{(n-1)}^{(2)} \mathcal{N}^{(2)} \hat{\pi}_1^{(2)} \text{or}_{(2)}^* X = \hat{\pi}_1^{(1)} \dots \hat{\pi}_1^{(n-1)} \mathcal{N}^{(n)} \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X .$$

Since  $\mathcal{Q}_{(n)} X := \hat{\pi}_1^{(1)} \dots \hat{\pi}_1^{(n)} \text{or}_{(n)}^* X$  and  $\text{or}_{(n)}^* X$  is  $(n, 2)$ -fibrant, in order to show (3.18) it suffices to show by induction on  $n \geq 2$  that

$$(3.20) \quad \mathcal{N}^{(n)} \hat{\pi}_1^{(1)} \dots \hat{\pi}_1^{(n)} Y = \hat{\pi}_1^{(1)} \dots \hat{\pi}_1^{(n-1)} \mathcal{N}^{(n)} \hat{\pi}_1^{(n)} Y$$

for any  $(n, 2)$ -fibrant  $n$ -fold simplicial set  $Y$ . For  $n = 2$ ,

$$\mathcal{N}^{(2)} \hat{\pi}_1^{(1)} \hat{\pi}_1^{(2)} Y = \hat{\pi}_1^{(1)} \mathcal{N}^{(2)} \hat{\pi}_1^{(2)} Y$$

by Lemma 1.18 and Proposition 1.17.

In the induction step, let  $G_\bullet$  be the simplicial  $(n-1)$ -fold groupoid  $\hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} Y$ . By Proposition 1.20,  $G_\bullet$  is  $(n-1, 2)$ -fibrant, so for each  $\vec{a} \in \Delta^{n-2}$ , the simplicial groupoid  $G_\bullet(\vec{a})$  (in the first groupoid direction of  $G_\bullet$ ) is  $(2, 2)$ -fibrant. Thus by Lemma 1.18 we have  $\mathcal{N}^{(n)} \hat{\pi}_1^{(1)} G_\bullet(\vec{a}) = \hat{\pi}_1^{(1)} \mathcal{N}^{(n)} G_\bullet(\vec{a})$ , so

$$\mathcal{N}^{(n)} \hat{\pi}_1^{(1)} \dots \hat{\pi}_1^{(n)} Y = \mathcal{N}^{(n)} \hat{\pi}_1^{(1)} G_\bullet = \hat{\pi}_1^{(1)} \mathcal{N}^{(n)} G_\bullet = \hat{\pi}_1^{(1)} \mathcal{N}^{(n)} \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} Y.$$

If we think of  $Y$  as a simplicial  $(n-1, 2)$ -fibrant  $(n-1)$ -fold simplicial set  $Y_\bullet^{(1)}$  (in the first direction), by the induction hypotheses

$$\mathcal{N}^{(n)} \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n)} Y_m^{(1)} = \hat{\pi}_1^{(2)} \dots \hat{\pi}_1^{(n-1)} \mathcal{N}^{(n)} \hat{\pi}_1^{(n)} Y_m^{(1)}$$

for each  $m \geq 0$ , so (3.20) holds for  $Y$ , too.

Observe that

$$(3.21) \quad \mathcal{N}^{(2)} \hat{\pi}_1^{(2)} \text{or}_{(2)}^* X = \mathcal{N} A^f$$

for  $A^f \in \mathbf{Gpd}(\mathcal{S})$  as in (2.3), where  $f : A \rightarrow B$  is the map of simplicial sets  $u : \text{Dec } X \rightarrow X$ . In fact,  $\hat{\pi}_1^{(2)} \text{or}_{(2)}^* X$ , thought of as a simplicial object in  $\mathbf{Gpd}$ , has  $(\hat{\pi}_1^{(2)} \text{or}_{(2)}^* X)_k = \hat{\pi}_1 \text{Dec}^k X$  in simplicial dimension  $k$ . This is isomorphic to the homotopically discrete groupoid  $(X_k)^{u_k}$  (where  $u_k : X_k \rightarrow X_{k-1}$  is a map of sets). Hence from (3.18) and (3.21) we conclude that

$$\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X = \overline{\mathcal{Q}}_{(n-1)} \mathcal{N} A^f .$$

Since  $(\mathcal{N} A^f)_k = \mathcal{L}_{(k+1)} X$  for each  $k \geq 0$ , (3.15) follows.

In particular, since  $\mathcal{Q}_{(n)} X \in \mathbf{Gpd}_t^n$ , we have, for  $k \geq 2$ ,

$$\mathcal{Q}_{(n-1)}(\mathcal{L}_{(k+1)} X) = (\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X)_k^{(n)} \cong (\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X)_1^{(n)} \times_{(\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X)_0^{(n)}} \dots \times_{(\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X)_1^{(n)}}^k (\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X)_1^{(n)}$$

so by (3.15) we have:

$$\mathcal{Q}_{(n-1)}(\mathcal{L}_{(k+1)} X) \cong (\mathcal{Q}_{(n-1)} \mathcal{L}_{(2)} X) \times_{(\mathcal{Q}_{(n-1)} \text{Dec } X)} \dots \times_{(\mathcal{Q}_{(n-1)} \text{Dec } X)}^k (\mathcal{Q}_{(n-1)} \mathcal{L}_{(2)} X) .$$

(b) By induction on  $n$ . For  $n = 1$ ,  $\mathcal{Q}_{(1)} = \hat{\pi}_1$ . Since by hypothesis  $X$  is homotopically discrete and  $u : \text{Dec } X \rightarrow X$  is a fibration,  $\mathcal{L}_{(2)} X = \text{Dec } X \times_X \text{Dec } X$  is

also homotopically discrete; hence  $\hat{\pi}_1 \mathcal{L}_{(2)} X$  is a homotopically discrete groupoid, and is therefore isomorphic to  $A^f$  where  $f : A \rightarrow B$  is the obvious map

$$X_1 \times_{X_0} X_1 \longrightarrow X_0 \times_{\pi_0 X} X_0 .$$

On the other hand,  $\hat{\pi}_1 \text{Dec } X \cong (X_1)^{d_0}$  and  $\hat{\pi}_1 X = (X_0)^\gamma$  (for  $\gamma : X_0 \rightarrow \pi_0 X$ ), so:

$$\hat{\pi}_1 \mathcal{L}_{(2)} X \cong \hat{\pi}_1 \text{Dec } X \times_{\hat{\pi}_1 X} \hat{\pi}_1 \text{Dec } X .$$

In the induction step, applying  $\mathcal{N}^{(n)}$  to both sides of (3.17), we must show that for each  $k \geq 2$  and  $i \geq 1$  we have

$$(\mathcal{N}^{(n)} \mathcal{Q}_{(n)}(\mathcal{L}_{(k)} X))_{i-1}^{(n)} \cong (\mathcal{N}^{(n)} \mathcal{Q}_{(n)} \text{Dec } X)_{i-1}^{(n)} \times_{(\mathcal{N}^{(n)} \mathcal{Q}_{(n)} X)_{i-1}^{(n)}} \cdots^k (\mathcal{N}^{(n)} \mathcal{Q}_{(n)} \text{Dec } X)_{i-1}^{(n)} ,$$

or equivalently (after applying (a)), that:

$$(3.22) \quad \mathcal{Q}_{(n-1)}(\mathcal{L}_{(i)}(\mathcal{L}_{(k)} X)) \cong \mathcal{Q}_{(n-1)} \mathcal{L}_{(i)} \text{Dec } X \times_{\mathcal{Q}_{(n-1)} \mathcal{L}_{(i)} X} \cdots^k \mathcal{Q}_{(n-1)} \mathcal{L}_{(i)} \text{Dec } X$$

Since  $X$  is homotopically discrete, so are  $\text{Dec } X$  and  $\mathcal{L}_{(k)} X$  (since  $u : \text{Dec } X \rightarrow X$  is a fibration), so we can apply induction hypothesis (b) for  $(n-1)$  to replace the left hand side of (3.22) by:

$$\mathcal{Q}_{(n-1)} \text{Dec}(\mathcal{L}_{(k)} X) \times_{\mathcal{Q}_{(n-1)} \mathcal{L}_{(k)} X} \cdots^i \times_{\mathcal{Q}_{(n-1)} \mathcal{L}_{(k)} X} \mathcal{Q}_{(n-1)} \text{Dec}(\mathcal{L}_{(k)} X) ,$$

and since  $\text{Dec}$  commutes with fiber products, and thus with  $\mathcal{L}_{(k)}$ , this equals:

$$(\mathcal{Q}_{(n-1)}(\text{Dec}^2 X \times_{\text{Dec } X} \cdots^k \text{Dec}^2 X)) \times_{(\mathcal{Q}_{(n-1)}(\text{Dec } X \times_X \cdots^k \text{Dec } X))} \cdots^i (\mathcal{Q}_{(n-1)}(\text{Dec}^2 X \times_{\text{Dec } X} \cdots^k \text{Dec}^2 X)) .$$

If we write  $A := \mathcal{Q}_{(n-1)} \text{Dec}^2 X$ ,  $B := \mathcal{Q}_{(n-1)} \text{Dec } X$ , and  $C := \mathcal{Q}_{(n-1)} X$ , applying (b) for  $n-1$  to this last expression yields:

$$(3.23) \quad (A \times_B \cdots^k \times_B A) \times_{(B \times_C \cdots^k \times_C B)} \cdots^i \times_{(B \times_C \cdots^k \times_C B)} (A \times_B \cdots^k \times_B A) .$$

Similarly, (b) applied to the right hand side of (3.22) yields

$$(3.24) \quad (A \times_B \cdots^i \times_B A) \times_{(B \times_C \cdots^i \times_C B)} \cdots^k \times_{(B \times_C \cdots^i \times_C B)} (A \times_B \cdots^i \times_B A) ,$$

and the two limits (3.23) and (3.24) are evidently equal.  $\square$

**3.25. Proposition.** *Let  $X$  be a Kan complex. Then:*

- (a) *The counit of the adjunction of §1.13 induces an  $n$ -equivalence  $X \rightarrow B\mathcal{Q}_{(n)} X$ .*
- (b)  *$\mathcal{Q}_{(n)}$  preserves weak equivalences of Kan complexes.*
- (c) *If  $X$  is homotopically discrete (i.e., all higher homotopy groups vanish), then  $\mathcal{Q}_{(n)} X$  is a homotopically discrete  $n$ -fold groupoid.*
- (d)  *$\mathcal{Q}_{(n)} X$  is an  $n$ -typical  $n$ -fold groupoid, and  $\Pi_0^{(n)} \mathcal{Q}_{(n)} X$  is isomorphic to  $\mathcal{Q}_{(n-1)} X$ .*

*Proof.* By induction on  $n$ . The claim is immediate for  $n = 1$  (with  $\mathcal{Q}_{(0)} X := \pi_0 X$ ).

(a) In the induction step, by Lemma 3.14,  $B\mathcal{Q}_{(n)}X$  is the realization of the simplicial space  $W$  with

$$W_k^v = \begin{cases} B\mathcal{Q}_{(n-1)} \text{Dec } X & \text{if } k = 0 \\ B\mathcal{Q}_{(n-1)}(\text{Dec } X \times_X \cdots \times_X \text{Dec } X) & \text{if } k \geq 1 \end{cases}$$

If we let  $Z \in [(\Delta^2)^{\text{op}}, \mathbf{Set}]$  denote the bisimplicial set with  $Z_k^v := \mathcal{L}_{(k+1)}X$  (cf. §3.11), then by induction, for each  $k \geq 0$  there is an  $(n-1)$ -equivalence  $W_k^v \simeq Z_k^v$ . Furthermore,  $W_0^v \simeq Z_0^v$  is actually a weak equivalence, hence in particular an  $n$ -equivalence. It follows from Proposition 2.24 that there is an  $n$ -equivalence  $B\mathcal{Q}_{(n)}X \simeq BW \simeq \text{Diag } Z$ .

Note that  $Z_k^h$  (i.e.,  $Z$  in vertical dimension  $k$ ) is weakly equivalent to the discrete simplicial set  $c(X_k)$ . It follows that  $\text{Diag } Z \simeq X$ , so in conclusion,  $X \rightarrow B\mathcal{Q}_{(n)}X$  is an  $n$ -equivalence.

(b) Let  $f : X \rightarrow Y$  be a weak equivalence of Kan complexes. Since, by (a),  $X \rightarrow B\mathcal{Q}_{(n)}X$  and  $Y \rightarrow B\mathcal{Q}_{(n)}Y$  are  $n$ -equivalences, it follows that  $B\mathcal{Q}_{(n)}f$  is an  $n$ -equivalence. By Theorem 3.7,  $B\mathcal{Q}_{(n)}X$  and  $B\mathcal{Q}_{(n)}Y$  are  $n$ -types. Hence  $B\mathcal{Q}_{(n)}f$  is a weak equivalence.

(c) Since  $X$  is homotopically discrete, by Lemma 3.14 for each  $k \geq 1$  we have:

$$(\mathcal{N}^{(n)}\mathcal{Q}_{(n)}X)_k = \mathcal{Q}_{(n-1)}(\mathcal{L}_{(k+1)}X) = \mathcal{Q}_{(n-1)}\text{Dec } X \times_{\mathcal{Q}_{(n-1)}X} \cdots \times_{\mathcal{Q}_{(n-1)}X} \mathcal{Q}_{(n-1)}\text{Dec } X.$$

Therefore  $\mathcal{Q}_{(n)}X = A^f$ , where  $A = \mathcal{Q}_{(n-1)}\text{Dec } X$  and by induction

$$f := \mathcal{Q}_{(n-1)}u : \mathcal{Q}_{(n-1)}\text{Dec } X \rightarrow \mathcal{Q}_{(n-1)}X$$

is a map of  $(n-1)$ -fold homotopically discrete groupoids. Hence, by definition,  $\mathcal{Q}_{(n)}X$  is homotopically discrete.

(d) By Lemma 3.14, for each  $1 \leq i \leq j \leq n$  and  $\vec{\mathbf{a}} \in \Delta^{n-2}$ , the double groupoid  $H := G(\vec{\mathbf{a}})$  is equal to  $(\mathcal{Q}_{(n-1)}Z_k)(\vec{\mathbf{a}})$  for some  $k \geq 0$  (where  $Z$  is the bisimplicial set defined above). Hence  $H$ , and so  $G$ , is symmetric.

To show that  $\mathcal{Q}_{(n)}X$  is  $n$ -typical, by Definition 2.19, we think of it as a groupoid  $(\mathcal{Q}_{(n)}X)_1 \xrightarrow{\rightarrow} (\mathcal{Q}_{(n)}X)_0$  in  $\mathbf{Gpd}^{n-1}$ . Note that  $(\mathcal{Q}_{(n)}X)_0 = \mathcal{Q}_{(n-1)}\text{Dec } X$ , and since  $\text{Dec } X$  is homotopically discrete,  $(\mathcal{Q}_{(n)}X)_0^{(n)}$  is homotopically discrete, by (c).

Similarly,  $(\mathcal{N}^{(n)}\mathcal{Q}_{(n)}X)_1^{(n)} = \mathcal{Q}_{(n-1)}\mathcal{L}_{(2)}X \in \mathbf{Gpd}_t^{n-1}$ , and by (3.16):

$$(\mathcal{Q}_{(n)}X)_1 \times_{(\mathcal{Q}_{(n)}X)_0} \cdots \times_{(\mathcal{Q}_{(n)}X)_0} (\mathcal{Q}_{(n)}X)_1 = \mathcal{Q}_{(n-1)}(\mathcal{L}_{(2)}X \times_{\text{Dec } X} \cdots \times_{\text{Dec } X} \mathcal{L}_{(2)}X),$$

so it is  $(n-1)$ -typical.

If we let  $\overline{\Pi}_0^{(n-1)}$  denote the result of applying  $\Pi_0^{(n-1)}$  in each simplicial dimension in the  $n$ -th direction, by Lemma 3.14(a) and the induction hypothesis:

$$(\overline{\Pi}_0^{(n-1)}\mathcal{N}^{(n)}\mathcal{Q}_{(n)}X)_k^{(n)} = \Pi_0^{(n-1)}\mathcal{Q}_{(n-1)}\mathcal{L}_{(k+1)}X = \mathcal{Q}_{(n-2)}\mathcal{L}_{(k+1)}X = (\mathcal{Q}_{(n-1)}X)_k.$$

where  $(\mathcal{N}^{(n-1)}\mathcal{Q}_{(n-1)}X)_k^{(n-1)}$  is abbreviated to  $(\mathcal{Q}_{(n-1)}X)_k$ .

This shows that  $\Pi_0^{(n)}G$  lands in  $(n-1)$ -typical  $(n-1)$ -fold groupoids, and that:

$$\Pi_0^{(n)}\mathcal{Q}_{(n)}X \cong \mathcal{Q}_{(n-1)}X .$$

To prove that  $\mathcal{Q}_{(n)}X \in \mathbf{Gpd}_t^n$ , it remains to show that in each simplicial dimension  $k \geq 0$  (in the  $n$ -th direction), we have:

$$(3.26) \quad (\mathcal{Q}_{(n)}X)_1 \times_{(\mathcal{Q}_{(n)}X)_0} \cdots \times_{(\mathcal{Q}_{(n)}X)_0}^k (\mathcal{Q}_{(n)}X)_1 \rightarrow (\mathcal{Q}_{(n)}X)_1 \times_{(\mathcal{N}^{(n)}\mathcal{Q}_{(n)}X)_0^d} \cdots \times_{(\mathcal{N}^{(n)}\mathcal{Q}_{(n)}X)_0^d}^k (\mathcal{Q}_{(n)}X)_1$$

is a weak equivalence. By Lemma 3.14 we have:

$$(\mathcal{Q}_{(n)}X)_1 \times_{(\mathcal{Q}_{(n)}X)_0} \cdots \times_{(\mathcal{Q}_{(n)}X)_0}^k (\mathcal{Q}_{(n)}X)_1 \cong \mathcal{Q}_{(n-1)}(\mathcal{L}_{(2)}X \times_{\mathrm{Dec} X} \cdots \times_{\mathrm{Dec} X}^k \mathcal{L}_{(2)}X)$$

and using Remark 1.23 for  $c(X_0)$  (the discrete simplicial set on  $X_0$ ), we have:

$$(\mathcal{Q}_{(n)}X)_1 \times_{(\mathcal{Q}_{(n)}X)_0^d} \cdots \times_{(\mathcal{Q}_{(n)}X)_0^d}^k (\mathcal{Q}_{(n)}X)_1 \cong \mathcal{Q}_{(n-1)}(\mathcal{L}_{(2)}X \times_{c(X_0)} \cdots \times_{c(X_0)}^k \mathcal{L}_{(2)}X) .$$

Since  $\mathcal{L}_{(2)}X \rightarrow \mathrm{Dec} X$  is a fibration and  $\mathrm{Dec} X \rightarrow c(X_0)$  is a weak equivalence, the map

$$\mathcal{L}_{(2)}X \times_{\mathrm{Dec} X} \cdots \times_{\mathrm{Dec} X}^k \mathcal{L}_{(2)}X \rightarrow \mathcal{L}_{(2)}X \times_{c(X_0)} \cdots \times_{c(X_0)}^k \mathcal{L}_{(2)}X$$

is a weak equivalence. Therefore, by (b), (3.26) is a weak equivalence, as required.  $\square$

Recall from §1.2 that  $P^n\mathcal{S}$  denotes the full subcategory of  $\mathcal{S} = [\Delta^{\mathrm{op}}, \mathbf{Set}]$  consisting of simplicial sets  $X$  for which the natural map  $X \rightarrow P^nX$  is a weak equivalence.

**3.27. Theorem.** *The functors  $\mathcal{Q}_{(n)} : P^n\mathcal{S} \rightarrow \mathbf{Gpd}_t^n$  and  $B : \mathbf{Gpd}_t^n \rightarrow P^n\mathcal{S}$  induce equivalences of categories after localizations*

$$\mathrm{ho}(P^n\mathcal{S}) \simeq \mathbf{Gpd}_t^n / \sim .$$

*Proof.* Let  $X \in \mathrm{ho}(P^n\mathcal{S})$ . We can assume without loss of generality that  $X$  is fibrant. By Theorem 3.7 and Proposition 3.25, there is a weak equivalence  $X \simeq B\mathcal{Q}_{(n)}X$  (so they are isomorphic in  $\mathrm{ho}(P^n\mathcal{S})$ ). Let  $G \in \mathbf{Gpd}_t^n / \sim$ . For the same reason, there is a weak equivalence  $BG \simeq B\mathcal{Q}_{(n)}BG$ , therefore  $G$  and  $\mathcal{Q}_{(n)}BG$  are isomorphic in the homotopy category  $\mathbf{Gpd}_t^n / \sim$ .

By definition,  $f_0 \sim f_1 : G \rightarrow H$  in  $\mathbf{Gpd}_t^n$  if and only if  $Bf_0 \sim Bf_1$  in  $\mathcal{S}$  – that is, if their geometric realizations  $|Bf_0| \sim |Bf_1|$  are homotopic maps of topological spaces. Since  $|BG|$  and  $|BH|$  are CW-complexes, this is the same as saying that  $[Bf_0] = [Bf_1]$  in  $\mathrm{ho}(P^n\mathcal{S})$  (see [Q1, I, §1]).  $\square$

**3.28. Remark.** There is an  $n$ -type model category on  $\mathcal{S}$  in which the weak equivalences are  $n$ -equivalences (see [H, §1.5 & Theorem 4.1.1] and compare [C]). Moreover, the functor  $\mathrm{or}_{(n)}^* : [\Delta^{\mathrm{op}}, \mathbf{Set}] \rightarrow [(\Delta^n)^{\mathrm{op}}, \mathbf{Set}]$  has a right adjoint (see Remark 1.23). We conjecture that this can be used to define a model category structure on  $\mathbf{Gpd}^n$ , in which the  $n$ -typical  $n$ -fold groupoids are fibrant, and extend Theorem 3.27 to a Quillen equivalence between this model category of  $n$ -fold groupoids and the  $n$ -type model category structure on  $\mathcal{S}$ .

#### 4. TAMSAMANI'S MODEL AND $n$ -TYPICAL $n$ -FOLD GROUPOIDS

In this section we construct a comparison functor from  $n$ -typical  $n$ -fold groupoids to Tamsamani's weak  $n$ -groupoids, which preserves homotopy types. As a result we obtain an alternative proof of Theorem 3.7.

**4.1. Tamsamani's weak  $n$ -groupoids.** We begin with a brief review of the definition of Tamsamani's weak  $n$ -groupoids. For full details, see [T], [Si], and [Pa]. The definition of the category  $\mathbf{Tam}^n$  of Tamsamani weak  $n$ -groupoids is by induction on  $n$ :

We set  $\mathbf{Tam}^1 = \mathbf{Gpd}$  and let  $\tau_0^{(1)} : \mathbf{Gpd} \rightarrow \mathbf{Set}$  be the functor  $\pi_0 \mathcal{N}$ ,  $\delta^{(1)} : \mathbf{Set} \rightarrow \mathbf{Gpd}$  the discrete groupoid functor and  $\tau_1^{(1)} : \mathbf{Tam}^1 \rightarrow \mathbf{Gpd}$  the identity functor. The 1-equivalences are the equivalences of groupoids.

Suppose we have inductively defined the subcategory  $\mathbf{Tam}^{n-1}$  of  $[\Delta^{\text{op}}, \mathbf{Tam}^{n-2}]$ , together with:

- (a) A distinguished class of morphisms, called  $(n-1)$ -equivalences.
- (b) A fully faithful and finite product-preserving functor  $\delta^{(n-1)} : \mathbf{Set} \rightarrow \mathbf{Tam}^{n-1}$ .
- (c) A functor  $\tau_0^{(n-1)} : \mathbf{Tam}^{n-1} \rightarrow \mathbf{Set}$  such that
  - i. The functor  $\delta^{(n-1)}$  is fully faithful and finite product-preserving, and  $\tau_0^{(n-1)} \delta^{(n-1)} \cong \text{Id}_{\mathbf{Set}}$ .
  - ii. The functor  $\tau_0^{(n-1)}$  sends  $(n-1)$ -equivalences to bijections. Objects in the image of  $\tau_0^{(n-1)}$  are called *discrete*.
  - iii. The functor  $\tau_0^{(n-1)}$  preserves fiber products over discrete objects.
  - iv. If  $f : X \rightarrow Y$  is a morphism in  $\mathbf{Tam}^{n-1}$  with  $Y$  discrete, there is an isomorphism  $X \cong \coprod_{y \in Y} f^{-1}\{y\}$ .

We now define  $\mathbf{Tam}^n$  to be the full subcategory of  $[\Delta^{\text{op}}, \mathbf{Tam}^{n-1}]$  consisting of simplicial objects  $X$  over  $\mathbf{Tam}^{n-1}$  satisfying the following conditions:

- (i)  $X_0$  is discrete;
- (ii) the *Segal maps*  $\mu_k : X_k \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$  (see [Se]) are  $(n-1)$ -equivalences for each  $k \geq 2$ .
- (iii) Define  $\bar{\tau}_0^{(n-1)} : [\Delta^{\text{op}}, \mathbf{Tam}^{n-1}] \rightarrow [\Delta^{\text{op}}, \mathbf{Set}]$  to be the functor  $\tau_0^{(n-1)}$  in each simplicial dimension. We see from (i) and (ii) that  $\bar{\tau}_0^{(n-1)} X$  is the nerve of a category. We require this category to be a groupoid, denoted by  $\tau_1^{(n)} X$ .

For any Tamsamani weak  $n$ -groupoid  $X$  and  $a, b \in X^0$ , let  $X(a, b)$  denote the fiber of  $(d_0, d_1) : X_1 \rightarrow X_0 \times X_0$  at  $(a, b)$ . We say that a map  $f : X \rightarrow Y$  in  $\mathbf{Tam}^n$  is an  $n$ -equivalence if:

- (i) For all  $a, b \in X_0$ ,  $f(a, b) : X(a, b) \rightarrow Y(f(a), f(b))$  is an  $(n-1)$ -equivalence.
- (ii)  $\tau_1^{(n)}$  is an equivalence of groupoids.

We set

$$(4.2) \quad \tau_0^{(n)} := \pi_0 \mathcal{N} \tau_1^{(n)}$$

and  $\delta^{(n)} = d\delta^{(n-1)}$ , where  $d : \mathbf{Tam}^{n-1} \rightarrow [\Delta^{\text{op}}, \mathbf{Tam}^{n-1}]$  is the constant simplicial object functor. It is easy to check that  $\delta^{(n)}$  and  $\tau_0^{(n)}$  satisfy (a)-(c) above. This completes the inductive definition of  $\mathbf{Tam}^n$ .

Note that there is an obvious embedding  $\mathbf{Tam}^n \hookrightarrow [(\Delta^{n-1})^{\text{op}}, \mathbf{Gpd}]$ . Composing this with  $\mathcal{N} : \mathbf{Gpd} \rightarrow [\Delta^{\text{op}}, \mathbf{Set}]$  and taking the  $n$ -fold diagonal  $\text{Diag}_n$  yields the *realization functor*  $B : \mathbf{Tam}^n \rightarrow [\Delta^{\text{op}}, \mathbf{Set}]$ .

**4.3. Lemma** ([Pa, Lemma 10.1]). *A map  $f : X \rightarrow Y$  in  $\mathbf{Tam}^n$  is an  $n$ -equivalence if and only if  $Bf : BX \rightarrow BY$  is a weak equivalence in  $[\Delta^{\text{op}}, \mathbf{Set}]$ .*

#### 4.4. Comparison with $n$ -typical $n$ -fold groupoids.

Let  $\mathcal{S}_h^2$  be the full subcategory of bisimplicial sets  $X$  such that the simplicial set  $X_0$  is homotopically constant, through a weak equivalence  $\gamma : X_0 \rightarrow X_0^d$  with a section  $\gamma' : X_0^d \rightarrow X_0$  with  $\gamma\gamma' = \text{Id}$ , where  $X_0^d$  is the constant simplicial set of  $\pi_0 X_0$ . Let  $\mathcal{S}_d^2$  denote the full subcategory of bisimplicial sets  $X$  such that the simplicial set  $X_0$  is constant. We construct a functor  $D : \mathcal{S}_h^2 \rightarrow \mathcal{S}_d^2$  as follows:

Given  $X \in \mathcal{S}_h^2$ ,

$$(DX)_n = \begin{cases} X_0^d & \text{if } n = 0 \\ X_n & \text{if } n > 0. \end{cases}$$

Let  $\partial_0, \partial_1 : X_1 \rightarrow X_0$  and  $\sigma_0 : X_0 \rightarrow X_1$  be the face and degeneracy maps of  $X$ , and let  $\partial'_0, \partial'_1 : X_1 \rightarrow X_0^d$ , and  $\sigma'_0 : X_0^d \rightarrow X_1$ , respectively, denote  $\partial'_i = \gamma\partial_i$  ( $i = 0, 1$ ), and  $\sigma'_0 = \sigma_0\gamma'$ . All other face and degeneracy operators of  $DX$  are the same as those of  $X$ .

**4.5. Lemma.** *Let  $D : \mathcal{S}_h^2 \rightarrow \mathcal{S}_d^2$  be as above. Then, for each  $X \in \mathcal{S}_h^2$ ,  $DX$  and  $X$  have the same homotopy type.*

*Proof.* We construct a bisimplicial set  $Y$  and weak equivalences  $X \xrightarrow{f} Y \xleftarrow{h} DX$ . Consider the pushout in  $[(\Delta^2)^{\text{op}}, \mathbf{Set}]$ :

$$\begin{array}{ccc} X_0 & \xrightarrow{s_{(n)}} & X_n \\ \gamma \downarrow & & \downarrow f_n \\ X_0^d & \xrightarrow{\sigma_{(n)}} & Y_n \end{array}$$

where  $s_{(n)}$  is induced by the unique morphism  $\mathbf{0} \rightarrow \mathbf{n}$  in  $\Delta^{\text{op}}$ . Since  $\gamma$  is a weak equivalence and  $s_{(n)}$  is a cofibration,  $f_n$  is a weak equivalence. Let  $\hat{\phi} : \mathbf{n} \rightarrow \mathbf{m}$  be any morphism in  $\Delta^{\text{op}}$ ; then  $\hat{\phi}s_{(n)} = s_{(m)}$  by the uniqueness, so that

$$f_m \hat{\phi}s_{(n)} = f_m s_{(m)} = \sigma_{(m)} f_0 : X_0 \rightarrow Y_m.$$

From the universal property of pushouts there exists a unique  $\hat{\phi} : Y_n \rightarrow Y_m$  with  $\hat{\phi}f_n = f_m \hat{\phi}$  and  $\hat{\phi}\sigma_{(n)} = \sigma_{(m)}$ . In particular, we have maps  $\hat{\partial}_i : Y_n \rightarrow Y_{n-1}$  for  $0 \leq i \leq n$ , and  $\hat{\sigma}_i : Y_{n-1} \rightarrow Y_n$  for  $0 \leq i < n$ . The maps  $\hat{\partial}_i$  and  $\hat{\sigma}_i$  satisfy the simplicial identities, so that  $Y_0$  is a simplicial object in  $[\Delta^{\text{op}}, \mathbf{Set}]$ . In fact, if  $\mathbf{n} \xrightarrow{\phi} \mathbf{m} \xrightarrow{\psi} \mathbf{k}$  are morphisms in  $\Delta^{\text{op}}$  and  $\xi = \hat{\phi} \circ \psi$ , then

$$\hat{\xi}\sigma_{(n)} = \sigma_{(k)} = \hat{\phi}_1\sigma_{(m)} = \hat{\phi}_1\hat{\phi}\sigma_{(n)}$$

and

$$\hat{\xi}f_n = f_k\xi = f_k\psi\hat{\phi} = \hat{\phi}'f_m\hat{\phi} = \psi\hat{\phi}f_n.$$



It follows by universal property of pushouts that  $\hat{\xi} = \hat{\psi}'\hat{\phi}$ . In particular, since the simplicial identities are satisfied by  $\partial_i$  and  $\sigma_i$ , they are satisfied by  $\hat{\partial}_i$  and  $\hat{\sigma}_i$ . So we have a map of bisimplicial sets  $f : X \rightarrow Y$  which is a levelwise weak equivalence; Therefore,  $Bf$  is also a weak equivalence.

Define  $h : DX \rightarrow Y$  by  $h_0 = \text{Id}$  and  $h_n = f_n$  for  $n > 0$ . It  $h$  is a map of bisimplicial sets. In fact, by construction,  $\partial'_i = \hat{\partial}_i f$ ; also,  $f_1 \sigma_0 = \hat{\sigma}_0 \gamma$ , which implies  $f_1 \sigma_0 \gamma' = \hat{\sigma}_0 \gamma \gamma' = \hat{\sigma}_0$ . All other identities are the same as for  $f$ . Since  $h$  is a levelwise weak equivalence,  $Bh$  is a weak equivalence. In conclusion,  $f$  and  $h$  are weak equivalences, so that  $\text{Diag } X \simeq \text{Diag } DX$ .  $\square$

**4.6. Definition.** We define the 0-*discretization* functor

$$\text{Disc}_0 : \mathbf{Gpd}_t^n \rightarrow [\Delta^{\text{op}}, \mathbf{Gpd}_t^{n-1}]$$

on any  $n$ -fold groupoid  $G$  as follows: set

$$(\text{Disc}_0 G)_n := \begin{cases} G_0^d & \text{if } n = 0 \\ (\mathcal{N}^{(1)}G)_n & \text{if } n > 0 \end{cases}$$

(cf. §1.10). If  $\partial_0, \partial_1 : G_1 \rightarrow G_0$  are the source and target maps, and  $\sigma_0 : G_0 \rightarrow G_1$  is the degeneracy operator (all in  $\mathbf{Gpd}^{n-1}$ ), we define  $\partial'_0, \partial'_1 : (\text{Disc}_0 G)_1 \rightarrow (\text{Disc}_0 G)_0$  and  $\sigma'_0 : (\text{Disc}_0 G)_0 \rightarrow (\text{Disc}_0 G)_1$  by  $\partial'_i = \gamma \partial_i$  ( $i = 0, 1$ ) and  $\sigma'_0 = \sigma_0 \gamma'$ . All other face and degeneracy operators of  $\text{Disc}_0 G$  are those of  $G$ . Since  $\gamma \gamma' = \text{Id}$ , all simplicial identities hold for  $\text{Disc}_0 G$ .

**4.7. Lemma.** For any  $n$ -typical  $n$ -fold groupoid  $G \in \mathbf{Gpd}_t^n$ ,  $BG$  and  $B \text{Disc}_0 G$  are weakly equivalent.

*Proof.*  $BG$  is the diagonal of the bisimplicial set  $X$  with  $X_k := B((\mathcal{N}^{(n)}G)_k^{(n)})$  for all  $k \geq 0$ , while  $B \text{Disc}_0 G$  is the diagonal of the bisimplicial set  $Y$  with  $Y_0 := G_0^d$  and  $Y_k := B((\mathcal{N}^{(n)}G)_k^{(n)})$  for  $k \geq 1$ . By construction,  $X \in \mathcal{S}_h^2$  and  $Y = DX$ . Hence, by Lemma 4.5,  $BG = \text{Diag } X \simeq \text{Diag } Y = B \text{Disc}_0 G$ .  $\square$

**4.8. Definition.** Let  $T_{(n)} : \mathbf{Gpd}_t^n \rightarrow \mathbf{Gpd}$  denote the composite

$$(4.9) \quad T_{(n)} := \Pi_0^{(2)} \cdots \Pi_0^{(n-1)} \Pi_0^{(n)}$$

(see Definitions 2.12 and 2.19). By construction,

$$(4.10) \quad (T_{(n)}G)_i = \pi_0 \mathcal{N} T_{(n-1)}(G_i)$$

for all  $i \geq 0$ .

Let  $D_1 : \mathbf{Gpd} \rightarrow \mathbf{Gpd}$  be the identity, and for  $n \geq 2$ , we define

$$D_n : \mathbf{Gpd}_t^n \rightarrow [(\Delta^{n-1})^{\text{op}}, \mathbf{Gpd}]$$

inductively to be the composite:

$$\mathbf{Gpd}_t^n \xrightarrow{\mathcal{N}^{(n)}} [\Delta^{\text{op}}, \mathbf{Gpd}_t^{n-1}] \xrightarrow{\text{Disc}_0} [\Delta^{\text{op}}, \mathbf{Gpd}_t^{n-1}] \xrightarrow{\overline{D}_{n-1}} [(\Delta^{n-1})^{\text{op}}, \mathbf{Gpd}]$$

where  $\overline{D}_{n-1}$  is obtained by applying  $D_{n-1}$  in each simplicial dimension.

**4.11. Proposition.** The functor  $D_n$  lands in  $\mathbf{Tam}^n$ . Furthermore,  $\tau_1^{(n)} D_n = T_{(n)}$ , and for each  $G \in \mathbf{Gpd}_t^n$ ,  $BG \simeq B D_n G$ .

*Proof.* By induction on  $n \geq 2$ . For  $n = 2$ , note  $D_2G = \text{Disc}_0 \mathcal{N}^{(2)}G$  is in  $\mathbf{Tam}^2$  for any 2-typical double groupoid  $G$ , since for each  $k \geq 2$  by Definition 2.19(iv) we have:

$$(D_2G)_k = G_1 \times_{G_0} \cdots \times_{G_0} G_1 \simeq G_1 \times_{G_0^d} \cdots \times_{G_0^d} G_1 \simeq (D_2G)_1 \times_{(D_2G)_0} \cdots \times_{(D_2G)_0} (D_2G)_1.$$

Furthermore,  $\tau_2^{(1)}D_2G = T_{(2)}G = \mathbf{\Pi}_0^{(2)}G$  is a groupoid. Hence, by definition,  $D_2G \in \mathbf{Tam}^2$ . By Lemma 4.7,  $BD_2G \simeq BG$  since  $G \in \mathbf{Gpd}_t^2$ .

In the induction step, note that  $(D_nG)_0 = G_0^d$  is discrete. So to prove that  $D_nG$  is in  $\mathbf{Tam}^n$ , it remains to show:

- (a) There are  $(n-1)$ -equivalences  $\mu_k : (D_nG)_k \rightarrow (D_nG)_1 \times_{(D_nG)_0} \cdots \times_{(D_nG)_0} (D_nG)_1$ .
- (b)  $\tau_1^{(n)}D_nG$  is a groupoid.

Note that by Definition 4.8 and by the inductive hypothesis, for  $k \geq 2$  we have:

$$(BD_nG)_k = BD_{n-1}(G_1 \times_{G_0} \cdots \times_{G_0} G_1) \simeq B(G_1 \times_{G_0} \cdots \times_{G_0} G_1),$$

and by Definition 2.19(iv) and the inductive hypothesis again this is weakly equivalent to

$$B(G_1 \times_{G_0^d} \cdots \times_{G_0^d} G_1) \simeq BD_{n-1}G_1 \times_{BG_0^d} \cdots \times_{BG_0^d} BD_{n-1}G_1$$

which is  $B((D_nG)_1 \times_{(D_nG)_0} \cdots \times_{(D_nG)_0} (D_nG)_1)$  by Definition 4.8. Thus each Segal map  $\mu_k$  is a weak equivalence, hence by Lemma 4.3 it is an  $(n-1)$ -equivalence. This proves (a).

To prove (b), note that by definition of  $\tau_1^{(n)}$ , (4.9), and (4.10), we have:

$$(\tau_1^{(n)}(D_nG))_0 = \tau_0^{(n-1)}(D_nG)_0 = \tau_0^{(n-1)}G_0^d = G_0^d = \pi_0 \mathcal{N}T_{(n-1)}(G_0) = (T_{(n)}G)_0.$$

Furthermore, since the discretization functor  $D_n$  does not affect simplicial dimensions  $\geq 1$ , we also have

$$(\tau_1^{(n)}(D_nG))_k = \tau_0^{(n-1)}(D_nG)_k = \tau_0^{(n-1)}D_{n-1}(\mathcal{N}^{(n)}G)_k$$

for  $k \geq 1$ . By induction we therefore have:

$$\begin{aligned} \tau_0^{(n-1)}D_{n-1}(\mathcal{N}^{(n)}G)_k &= \pi_0 \mathcal{N} \tau_1^{(n-1)}(D_{n-1}(\mathcal{N}^{(n)}G))_k \\ &= \pi_0 \mathcal{N}(T_{(n-1)}\mathcal{N}^{(n)}G)_k = (T_{(n)}\mathcal{N}^{(n)}G)_k, \end{aligned}$$

using (4.2), and the fact that  $\pi_0 \mathcal{N} = \mathbf{\Pi}_0^{(2)}$ .

It follows that  $\tau_1^{(n)}D_nG = T_{(n)}G$ , as claimed. Since  $T_{(n)}G$  is a groupoid, so is  $\tau_1^{(n)}D_nG$ . This concludes the proof that  $D_nG \in \mathbf{Tam}^n$ .

Finally, we show that  $BD_nG \simeq BG$ . Let  $Y = \text{Disc}_0 \mathcal{N}^{(n)}G \in [\Delta^{\text{op}}, \mathbf{Gpd}_t^{n-1}]$ . By Lemma 4.7,  $BY \simeq BG$ . Furthermore,  $BD_nG$  is the realization of the bisimplicial set  $Z$  with  $Z_k := BD_{n-1}Y_k$ . By induction,  $Z_k \simeq BY_k$ , so that  $\text{Diag } Z \simeq BY \simeq BG$ , as required.  $\square$

**4.12. Remark.** Since by [T],  $BD_nG$  is an  $n$ -type, Proposition 4.11 implies that the realization of an  $n$ -typical  $n$ -fold groupoid is an  $n$ -type. This provides an alternative

proof of the first statement in Theorem 3.7. Moreover, [T, §5] provides a formula for the homotopy groups:

$$\pi_n(BD_n G, x) = \text{Aut}_{\mathcal{C}_n(D_n G)}(\text{Id}_x) ,$$

where  $\mathcal{NC}_n(D_n G)$  is  $D_n \mathcal{W}_{(n, n-1)} G$ . This matches (3.8).

**4.13. Weakly globular  $n$ -fold groupoids.** One can define *weakly globular  $n$ -fold groupoids* as in Definition 2.19, with a specified ordering of the groupoid directions, but without requiring symmetry (for  $n = 2$ , see [BP, §2.19]).

We then have functors  $\Pi_0^{(n)}$  and natural transformations  $\gamma_n$  as in §2.19, and any homotopically discrete  $n$ -fold groupoid is weakly globular. Note that any strict  $n$ -groupoid  $G$  is in particular weakly globular, with the  $(n - k - 1)$ -fold groupoids of objects of  $\mathcal{W}_{(n, k)} G$  all discrete.

In fact, it can be shown that Theorem 3.7 and Proposition 4.11 hold more generally for any weakly globular  $n$ -fold groupoid, so the latter also model all  $n$ -types. A similar result for path connected  $n$ -types appears in [Pa].

## 5. APPLICATIONS AND FURTHER DIRECTIONS

In this section we provide an application for our model of  $n$ -types, and indicate some directions for future work.

### 5.1. $(k - 1)$ -connected $n$ -types.

Our first application is to provide an algebraic model of  $(k - 1)$ -connected  $n$ -types and relate it to the homotopy types of iterated loop spaces. This was mentioned in [BD] as a desirable feature for models of  $n$ -types (see also [Be]).

Recall that a space  $X$  is  $(k - 1)$ -connected if  $\pi_0 X = 0$  and  $\pi_i(X, x) = 0$  for  $1 \leq i \leq k - 1$ , and all  $x \in X$ . We denote the category of  $(k - 1)$ -connected  $n$ -types by  $\mathcal{P}_k^n \mathcal{S}$ .

**5.2. Lemma.** *If  $X$  is a  $(k - 1)$ -connected Kan complex,  $X$  is weakly equivalent to a  $(k - 1)$ -reduced Kan complex  $\hat{X}$  – that is,  $\hat{X}_i = \{*\}$  for  $1 \leq i \leq k - 1$ .*

*Proof.* See [GJ, III, §3]. □

**5.3. Definition.** A homotopically discrete  $n$ -fold groupoid  $G$  is *contractible* if  $\pi_0 BG$  is trivial (so that  $BG$  is contractible).

More generally, an  $n$ -typical  $n$ -fold groupoid  $G$  is called  $(n, k)$ -typical if for each  $0 \leq r < k$ , the homotopically discrete  $(n - r - 1)$ -fold groupoid

$$G_{1 \dots 10}^{(n-r \dots n)} = (\mathcal{W}_{(n, k)} G)_0^{(n-r)}$$

is contractible. This is the  $(n - r - 1)$ -fold groupoid of objects of the  $(n - r)$ -fold groupoid  $\mathcal{W}_{(n, k)} G \in \mathbf{Gpd}_t^{n-r}$  (see §2.21).

In particular, when  $r = 0$ , this just means that the  $(n - 1)$ -fold groupoid of objects  $G_0^{(n)}$  of  $G$  in the  $n$ -th direction (which is a homotopically discrete  $(n - 1)$ -fold groupoid), is contractible.

We let  $\mathbf{Gpd}_t^{(n, k)}$  denote the full subcategory of  $(n, k)$ -typical  $n$ -fold groupoids in  $\mathbf{Gpd}_t^n$ .

We now want to show that  $\mathbf{Gpd}_t^{(n,k)}$  is an algebraic model of  $(k-1)$ -connected  $n$ -types. For this, we need the following:

**5.4. Lemma.** *If  $X$  is a  $(k-1)$ -reduced Kan complex, then  $\mathcal{Q}_{(n)}X$  is  $(n,k)$ -typical.*

*Proof.* By Lemma 3.14(a),  $(\mathcal{Q}_{(n)}X)_0^{(n)} = \mathcal{Q}_{(n-1)} \text{Dec } X$ . Since  $\text{Dec } X \simeq c(X_0) = *$  and  $\mathcal{Q}_{(n-1)}$  preserves weak equivalences of Kan complexes by Proposition 3.25(b), we have  $\mathcal{Q}_{(n-1)} \text{Dec } X \simeq \mathcal{Q}_{(n-1)}(*) = *$ . Therefore,  $B((\mathcal{Q}_{(n)}X)_0)$  is contractible.

We now show by induction on  $1 \leq r < k$  that

$$(5.5) \quad \mathcal{W}_{(n,r)}(\mathcal{Q}_{(n-1)}X) := (\mathcal{N}^{(n-r+1, \dots, n)} \mathcal{Q}_{(n)}X)_{1 \dots 1}^{(n-r+1, \dots, n)} = \mathcal{Q}_{(n-r)} \mathcal{L}_{(2)}^r X$$

(in the notation of (3.12), where  $\mathcal{L}_{(2)}^r X := \mathcal{L}_{(2)}^{r-1}(\mathcal{L}_{(2)}X)$ .) The case  $r = 1$  is Lemma 3.14(a) for  $k = 1$ , which implies that we have an isomorphism of  $(n-1)$ -fold groupoids:

$$(5.6) \quad \mathcal{W}_{(n,1)}(\mathcal{Q}_{(n-1)}X) := (\mathcal{N}^{(n)} \mathcal{Q}_{(n)}X)_1^{(n)} \cong \mathcal{Q}_{(n-1)}(\mathcal{L}_{(2)}X) .$$

In the induction step, since  $\mathcal{L}_{(2)}X$  is still a Kan complex, by (2.22) we can apply the induction hypothesis to right hand side of (5.6) to deduce that:

$$\mathcal{W}_{(n,r)}(\mathcal{Q}_{(n-1)}\mathcal{L}_{(2)}X) \cong \mathcal{Q}_{(n-r)}(\mathcal{L}_{(2)}^{r-1}\mathcal{L}_{(2)}X) ,$$

which yields (5.5). From this and Lemma 3.14(a) (for  $k = 0$ ) we have

$$(\mathcal{W}_{(n,r)}\mathcal{Q}_{(n)}X)_0 = \mathcal{Q}_{(n-r-1)} \text{Dec } \mathcal{L}_{(2)}^r X ,$$

and since  $\text{Dec } \mathcal{L}_{(2)}^r X \simeq c(\mathcal{L}_{(2)}^r X)_0$ , we have  $B((\mathcal{W}_{(n,r)}\mathcal{Q}_{(n)}X)_0) \simeq c(\mathcal{L}_{(2)}^r X)_0$ .

Note that since  $X$  is  $(k-1)$ -reduced,  $\text{Dec } X$ , and thus  $\mathcal{L}_{(2)}X$ , are  $(k-2)$ -reduced, so by induction  $\mathcal{L}_{(2)}^r X$  is  $(k-r-1)$ -reduced. Thus as long as  $r < k$ ,  $\mathcal{L}_{(2)}^r X$  is 0-reduced, so  $B((\mathcal{W}_{(n,r)}\mathcal{Q}_{(n)}X)_0)$  is contractible.  $\square$

**5.7. Theorem.** *The functor  $\mathcal{Q}_{(n)}$  induces an equivalence of categories:*

$$\text{ho } \mathcal{P}_k^n \mathcal{S} \approx \mathbf{Gpd}_t^{(n,k)} / \sim .$$

*Proof.* If  $X \in \text{ho}(\mathcal{P}_k^n \mathcal{S})$ , we can assume without loss of generality that  $X_i = \{*\}$  for  $0 \leq i \leq k-1$ . Then, by Lemma 5.4,  $\mathcal{Q}_{(n)}X \in \mathbf{Gpd}_t^{(n,k)}$ . The result then follows immediately from Theorem 3.27.  $\square$

Note that the composition of  $\mathcal{W}_{(n,k)}$  of §2.21 with the classifying space functor  $B$  lands in  $\mathcal{P}^n \mathcal{S}$ , by Theorem 3.27, so its restriction to  $\mathbf{Gpd}_t^{(n,k)}$  lands in the category  $\mathcal{P}^{n-k} \mathcal{S}$  of  $(n-k)$ -types. Let  $\mathcal{P}_{\Omega^k}^{n-k}$  denote the subcategory of  $(n-k)$ -type  $k$ -fold loop spaces. We now show:

**5.8. Theorem.** *The restriction of  $B\mathcal{W}_{(n,k)}$  to  $\mathbf{Gpd}_t^{(n,k)}$  lifts to a functor  $\mathbf{Gpd}_t^{(n,k)} \rightarrow \mathcal{P}_{\Omega^k}^{n-k}$ , which induces an equivalence of categories:*

$$\mathbf{Gpd}_t^{(n,k)} / \sim \approx \text{ho } \mathcal{P}_{\Omega^k}^{n-k}$$

*Proof.* We shall show by induction on  $k$  that  $B\mathcal{W}_{(n,k)}G \simeq \Omega^k BG$  for any  $G \in \mathbf{Gpd}_t^{(n,k)}$ .

For  $k = 1$ , consider the simplicial  $(n - 1)$ -fold groupoid  $\mathcal{N}^{(n)}G_\bullet^{(n)}$ . Applying the classifying space functor in each simplicial dimension yields a bisimplicial set  $Y_\bullet = B\mathcal{N}^{(n)}G_\bullet^{(n)}$ . Thus  $Y_0 = B((\mathcal{N}^{(n)}G)_0^{(n)})$  is contractible, and the Segal maps for  $Y_\bullet$  are isomorphisms (since  $\mathcal{N}^{(n)}G_\bullet^{(n)}$  is the nerve of an internal groupoid), hence in particular weak equivalences.

As  $G$  is  $n$ -typical, applying the functor  $T_{(n)}$  of §4.8 yields a groupoid, and  $\pi_0 Y_\bullet = \mathcal{N}T_{(n)}G$ . Since  $Y_0$  is contractible,  $\pi_0 Y_\bullet$  is the nerve of a group. Thus  $Y_1$  has a homotopy inverse (cf. [Do, (6.3,4)]), so it follows from [Se, Proposition 1.5] that  $BY_1 \simeq \Omega BY_\bullet$ . That is,

$$B(G_1^{(n)}) = B\mathcal{W}_{(n,1)}G \simeq \Omega BG.$$

In the induction step, let

$$H := \mathcal{W}_{(n,1)}G = (\mathcal{N}^{(n)}G)_1^{(n)}$$

in  $\mathbf{Gpd}_t^{(n-1,k-1)}$ , where by hypothesis  $B\mathcal{W}_{(n-1,k-1)}H \simeq \Omega^{k-1}BH$ . By what we have shown above for  $k = 1$  we have  $BH = \mathcal{N}^{(n)}G_1^{(n)} \simeq \Omega BG$ . It follows that

$$B\mathcal{W}_{(n,k)}G = B\mathcal{W}_{(n-1,k-1)}H \simeq \Omega^{k-1}BH \simeq \Omega^{k-1}(\Omega BG) = \Omega^k BG.$$

Let  $E_{(k)} : \mathcal{P}_{\Omega^k}^{(n-k)} \rightarrow \mathcal{P}_k^n$  be the  $k$ -fold delooping functor of [M, Theorem 13.1]. To each  $(n - k)$ -type  $k$ -fold loop space  $Y = \Omega^k X \in \mathcal{P}_{\Omega^k}^{(n-k)}$  we associate the  $(n, k)$ -typical  $n$ -fold groupoid  $\mathcal{Q}_{(n)}E_{(k)}Y \in \mathbf{Gpd}_t^{(n,k)}$ . By what we have shown we have:

$$\mathcal{W}_{(n,k)}(B\mathcal{Q}_{(n)}E_{(k)}Y) \simeq \Omega^k(\mathcal{Q}_{(n)}E_{(k)}Y) \simeq \Omega^k E_{(k)}Y = \Omega^k E_{(k)}\Omega^k X \simeq \Omega^k X = Y.$$

Conversely, given  $\mathcal{G} \in \mathbf{Gpd}_t^{(n,k)}$ , we have

$$B\mathcal{Q}_{(n)}E_{(k)}\mathcal{W}_{(n,k)}G \simeq E_{(k)}\Omega^k BG \simeq BG,$$

which completes the proof.  $\square$

### 5.9. Further directions.

As stated in the Introduction, our main motivation in constructing our model for  $n$ -types was to obtain useful algebraic approximations of homotopy theories – that is, of simplicially enriched categories.

Recall that if  $\langle \mathcal{V}, \otimes, I \rangle$  is any monoidal category, we denote by  $\mathcal{V}\text{-}\mathbf{Cat}$  the collection of all (not necessarily small)  $\mathcal{V}$ -categories, that is, categories enriched in  $\mathcal{V}$  (see [Bo, Vol. II, §6.2]). We obtain further variants by applying any (strictly) monoidal functor  $P : \langle \mathcal{V}, \otimes \rangle \rightarrow \langle \mathcal{V}', \otimes' \rangle$  to a  $\mathcal{V}$ -category  $\mathcal{C}$ . For example, given an  $\mathcal{S}$ -category  $X_\bullet$ , for each  $n \geq 1$  we have a  $P^n\mathcal{S}$ -category  $Y_\bullet := P^n X_\bullet$ , in which each mapping space  $Y_\bullet(a, b)$  is the  $n$ -th Postnikov section  $P^n X_\bullet(a, b)$ .

**5.10.  $n$ -Track categories.** For  $n \geq 2$ , an  $n$ -track category is a category enriched in  $n$ -typical  $n$ -fold groupoids  $(\mathbf{Gpd}_t^n, \times)$ , with respect to the cartesian monoidal structure. The category of  $n$ -track categories is denoted by  $\mathbf{Track}_n$ .

Since  $\mathcal{Q}_{(n)} : [\Delta^{\text{op}}, \mathbf{Set}] \rightarrow \mathbf{Gpd}_t^n$  preserves products (see §1.23), it induces a functor

$$S_{(n)} : \mathcal{S}\text{-Cat} \longrightarrow \mathbf{Track}_n$$

from simplicial categories to  $n$ -track categories. Furthermore, the functors  $\Pi_0^{(n)} : \mathbf{Gpd}_t^n \rightarrow \mathbf{Gpd}_t^{n-1}$  giving the Postnikov decomposition of  $\mathbf{Gpd}_t^n$  induce functors

$$P^{n-1} : \mathbf{Track}_n \rightarrow \mathbf{Track}_{n-1}$$

providing the Postnikov decomposition of simplicially enriched categories.

For  $n = 1$ , the corresponding  $k$ -invariant was described in [BW] in terms of the Baues-Wirsching cohomology of categories, and a similar result was obtained in [BP] for  $n = 2$ , using an algebraically-defined cohomology of track categories. The extension of this to general  $n$  via an appropriate cohomology of  $(n - 1)$ -track categories will be the subject of a subsequent paper.

**5.11. Spectral sequences.** In [BB1], the authors introduced the notion of the *Postnikov  $n$ -stem*  $\mathcal{P}[n]X$  of a topological space  $X$  – that is, the system of  $(k - 1)$ -connected  $(n + k)$ -Postnikov sections  $P^{n+k}X\langle k - 1 \rangle$  ( $k = 0, 1, \dots$ ), with the natural maps between them.

They then show that the  $E^{n+2}$ -term of the homotopy spectral sequence of a (co)simplicial space  $W_\bullet$  (respectively,  $W^\bullet$ ) depends only on the simplicial  $n$ -stems  $\mathcal{P}[n]W_\bullet$  or  $\mathcal{P}[n]W^\bullet$ . Thus we can in principle use the  $(n + k)$ -fold groupoid models of each  $W_m$  or  $W^m$ , as in §5.1 to compute the  $d^{n+1}$ -differentials.

However, in many cases of interest – including the (stable or unstable) Adams spectral sequence, the Eilenberg-Moore spectral sequence, and others – a more “algebraic” approach can be used, using the notion of  *$n$ -th order derived functors* introduced in [BB3].

For example, the (unstable)  $\mathbb{F}_p$ -Adams spectral sequence for a (simply connected) space  $X$  constructed in [BK] is the homotopy spectral sequence of a cosimplicial space  $W^\bullet$  obtained as a  $\mathbb{F}_p$ -resolution of  $X$ . It can be shown that the  $E^{n+2}$ -term of this spectral sequence depends only on the  $n$ -Postnikov sections of the mapping spaces  $\text{map}(X, E)$  and  $\text{map}(E, E')$  for various products of  $\mathbb{F}_p$ -Eilenberg-Mac Lane space  $E$  and  $E'$ . Thus we do not need a full algebraic model for the  $P^n\mathcal{S}$ -category  $\mathbf{Top}$ , but only for the small subcategory with objects  $X$  and  $E$  as above. Since all mapping spaces in this category are themselves simplicial  $\mathbb{F}_p$ -vector spaces, the associated  $n$ -track category is correspondingly simplified. The case  $n = 1$  was treated in great detail in [Ba2].

**5.12.  $\infty$ -groupoids.** There are several groupoid-based models for general (non-truncated) homotopy types, starting from the simplicial groupoids of [DK3], and including the Segal groupoids of [Be3], the  $\omega$ - and  $\infty$ -groupoids of [Br, BH, KV1, KV2], and others.

For any Kan complex  $X$  and  $n \geq 1$ , by Proposition 3.25(d) we have a natural isomorphism of  $(n - 1)$ -typical  $(n - 1)$ -fold groupoids  $\Pi_0^{(n)}\mathcal{Q}_{(n)}X \xrightarrow{\cong} \mathcal{Q}_{(n-1)}X$ , and by Lemma 2.25 there is a natural  $(n - 1)$ -equivalence  $\mathcal{Q}_{(n)}X \rightarrow c\Pi_0^{(n)}\mathcal{Q}_{(n)}X$ . Composing these two maps yields a natural  $(n - 1)$ -equivalence  $\rho_n : \mathcal{Q}_{(n)}X \rightarrow \mathcal{Q}_{(n-1)}X$ , and we set  $\mathcal{Q}_{(\infty)}X$  to be the pro-object

$$(5.13) \quad \dots \mathcal{Q}_{(n)}X \rho_n : \xrightarrow{\rho_n} \mathcal{Q}_{(n-1)}X \xrightarrow{\rho_{n-1}} \dots \mathcal{Q}_{(1)}X := \hat{\pi}_1 X .$$

Applying the functor  $B$  to (5.13) and taking the homotopy limit recovers  $X$  up to weak equivalence, by Proposition 3.25(e), so we may call  $\mathcal{Q}_{(\infty)}X$  an  $\infty$ -typical  $\infty$ -groupoid model of  $X$ .

We observe that the symmetry of an  $n$ -typical  $n$ -fold groupoid  $G$  allows one to describe it more succinctly by a diagram of sets:

$$(5.14) \quad \widehat{G}_n \rightrightarrows \widehat{G}_{n-1} \rightrightarrows \widehat{G}_{n-2} \rightrightarrows \dots \widehat{G}_1 \rightrightarrows \widehat{G}_0$$

where  $\widehat{G}_k := G_{1\dots 10 \dots 0, \text{ say. } k \text{ } n-k}$ . Of course, this notation omits the composition maps, as well as the various automorphisms of the sets  $\widehat{G}_k$ , allowing one to recover the source and target maps in the other directions.

Moreover, when  $G = G^{(n)} := \mathcal{Q}_{(n)}X$ , we can recover  $G^{(n-1)} := \mathcal{Q}_{(n-1)}X \cong \Pi_0^{(n)}\mathcal{Q}_{(n)}X$  by taking coequalizers in the vertical direction:

$$\begin{array}{ccc} \widehat{G}_n^{(n)} & \xrightleftharpoons[t]{s} & \widehat{G}_{n-1}^{(n)} & \xrightleftharpoons[t]{s} & \dots & \widehat{G}_2^{(n)} & \xrightleftharpoons[t]{s} & \widehat{G}_1^{(n)} \\ \downarrow s \quad \downarrow t & & \downarrow s \quad \downarrow t & & & \downarrow s \quad \downarrow t & & \downarrow s \quad \downarrow t \\ \widehat{G}_{n-1}^{(n)} & \xrightleftharpoons[t]{s} & \widehat{G}_{n-2}^{(n)} & \xrightleftharpoons[t]{s} & \dots & \widehat{G}_1^{(n)} & \xrightleftharpoons[t]{s} & \widehat{G}_0^{(n)} \\ \downarrow q_{n-1}^{(n)} & & \downarrow q_{n-2}^{(n)} & & & \downarrow q_1^{(n)} & & \downarrow q_0^{(n)} \\ \widehat{G}_{n-1}^{(n-1)} & \xrightleftharpoons[t]{s} & \widehat{G}_{n-2}^{(n-1)} & \xrightleftharpoons[t]{s} & \dots & \widehat{G}_1^{(n-1)} & \xrightleftharpoons[t]{s} & \widehat{G}_0^{(n-1)} \end{array}$$

In this description,  $\mathcal{Q}_{(\infty)}X$  in groupoid dimension  $k$  may be thought of as a tower of sets:  $\dots \widehat{G}_k^{(n)} \xrightarrow{q_k^{(n)}} \widehat{G}_k^{(n-1)} \dots \widehat{G}_k^{(0)}$ .

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